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Colouring the Real Line

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The problem of colouring the real line so that the distance between like coloured numbers does not lie in some specified set D, called the *distance set*, is discussed. In particular, the minimum number of colours needed for various distance sets are determined. © 1985 Academic Press, Inc.

1. INTRODUCTION

What is the least number of colours which can be used to colour all points of the euclidean plane so that no two points which are unit distance apart have the same colour? Though rather well known, this problem has resisted all attempts at solution. The necessity of four colours was established by Moser and Moser [2], and the sufficiency of seven colours by Hadwiger, Debrunner, and Klee [1]. Essentially no further progress has been made on the problem.

The corresponding problem for the real line is easy: there are various ways in which two colours can be used to colour the line so that points which are unit distance apart have different colours. But how many colours are needed to avoid assigning the same colour to points whose distance apart lies between $1-\varepsilon$ and $1+\varepsilon$ (where $0<\varepsilon<1$)? We shall discuss this and several related problems. By scaling, the closed interval $[1-\varepsilon, 1+\varepsilon]$ can be converted into the closed interval $[1, \delta]$ for suitable $\delta>1$, so we shall in fact treat this "normalized" version of the problem.

2. SOME DEFINITIONS

A *proper colouring* of a graph is a colouring of the vertices (i.e., an assignment of each vertex of the graph to a colour class) so that no two

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adjacent vertices have the same colour. The *chromatic number* of the graph is the least number of colour classes which admits a proper colouring of the graph.

Given any set D of positive real numbers, let $G(\mathbb{R}, D)$ denote the graph whose vertices are all the points of the real line $\mathbb{R}^1$, such that any two points x, y are adjacent if and only if $|x-y| \in D$. The set D will be called the *distance set* of the graph. We shall treat the problem of determining the chromatic number of $G(\mathbb{R}, D)$ for various classes of distance set D.

3. Closed Interval of Distances

The first case to be considered is when the distance set D is a *closed interval* $[1, \delta]$ for some $\delta \geqslant 1$. For convenience, in this case we shall use $\mathbb{R}(\delta)$ to denote the graph $G(\mathbb{R}, D)$, and $\chi(\delta)$ to denote its chromatic number.

Lemma 1. *If* $1 \leqslant \delta \leqslant n$ *then* $\chi(\delta) \leqslant n+1$, *for any positive integer* n.

Proof. Let all points of the real line be assigned to colour classes numbered $0, 1, \ldots, n$ such that x has colour i exactly when $\lfloor x \rfloor \equiv i \pmod{n+1}$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. (Fig. 1). This is a proper colouring of $\mathbb{R}(\delta)$ if $1 \leqslant \delta \leqslant n$, whence $\chi(\delta) \leqslant n+1$. ∎

Theorem 1. *If* $\delta \geqslant 1$ *and* $n-1 < \delta \leqslant n$ *then* $\chi(\delta) = n+1$, *for any positive integer* n.

Proof. The definition of $\chi(\delta)$ requires that $\delta \geqslant 1$; the case $n=1$ corresponds to $\delta = 1$. Obviously $\mathbb{R}(1)$ needs at least two colours for a proper colouring, and Lemma 1 shows that it can be properly coloured with two colours, so $\chi(1) = 2$.

Now suppose $n \geqslant 2$ and $n-1 < \delta \leqslant n$, and consider a proper colouring of $\mathbb{R}(\delta)$. For any $\varepsilon > 0$ we can choose vertices x, y which are coloured differently and satisfy $0 < y - x < \varepsilon$. In particular, choose $\varepsilon < 1 + \delta - n$. Define vertices $v_i := y + i$ for $1 \leqslant i \leqslant n-1$. Now $v_{n-1} - y = n - 1 < \delta$ so

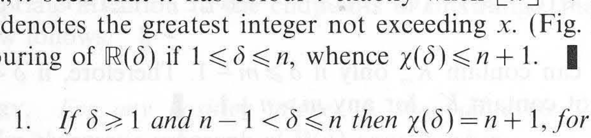

Fig. 1. A proper colouring of $\mathbb{R}(\delta)$, for $1 \leqslant \delta \leqslant n$.

$v_{n-1}-x<v_{n-1}-y+\varepsilon<\delta$. Thus, apart from x and y, the distance between any two points in $\{x, y, v_1, v_2, ..., v_{n-1}\}$ is at least 1 but less than δ, so all such pairs are adjacent in $\mathbb{R}(\delta)$. Hence $x, y, v_1, v_2, ..., v_{n-1}$ must all have different colours in the chosen proper colouring of $\mathbb{R}(\delta)$, and so $\chi(\delta) \geqslant n+1$. Equality then follows from Lemma 1. ∎

What subgraphs of $\mathbb{R}(\delta)$ are "responsible" for its chromatic number? We define a *chromatic subgraph* of a graph G to be a minimal subgraph of G with the same chromatic number as G. A graph G is *colour-critical* if its only chromatic subgraph is G itself. The next three theorems are concerned with chromatic subgraphs of $\mathbb{R}(\delta)$. We begin by investigating when a complete graph is a chromatic subgraph of $\mathbb{R}(\delta)$.

THEOREM 2. *If $n \leqslant \delta < n+1$, the largest complete subgraph of $\mathbb{R}(\delta)$ is K_{n+1}, for any positive integer n.*

Proof. If $\delta \geqslant n$, the vertices 0, 1, 2,..., n in $\mathbb{R}(\delta)$ induce the complete subgraph K_{n+1}. On the other hand, if $\mathbb{R}(\delta)$ contains the complete subgraph K_m with vertices $v_1 > v_2 > \cdots > v_m$ then

$$m-1 \leqslant (v_1-v_2)+(v_2-v_3)+\cdots+(v_{m-1}-v_m)=v_1-v_m \leqslant \delta.$$

Hence $\mathbb{R}(\delta)$ can contain K_m only if $\delta \geqslant m-1$. Therefore, if $\delta < n+1$ then $\mathbb{R}(\delta)$ does not contain K_m for any $m > n+1$. ∎

COROLLARY. *The complete graph K_{n+1} is the smallest order chromatic subgraph of $\mathbb{R}(\delta)$ exactly when $\delta = n$, for any positive integer n.*

Next we investigate when an odd cycle is a chromatic subgraph of $\mathbb{R}(\delta)$.

THEOREM 3. *The smallest odd cycle in $\mathbb{R}(\delta)$ is C_3 if $\delta \geqslant 2$, and C_{2n+1} if $1+1/n \leqslant \delta < 1+1/(n-1)$, for any integer $n \geqslant 2$.*

Proof. Theorem 2 implies that C_3 (which is K_3) is a subgraph of $\mathbb{R}(\delta)$ exactly when $\delta \geqslant 2$.

Suppose $\mathbb{R}(\delta)$ contains an odd cycle $C_{2n+1} := v_0 v_1 \cdots v_{2n} v_0$. Since $(v_0-v_1)+(v_1-v_2)+\cdots+(v_{2n-1}-v_{2n})+(v_{2n}-v_0)=0$, the sum of the signed distances corresponding to the edges of this cycle is zero. Since the cycle can be reflected about any point, we can assume without loss of generality that at least $n+1$ of the signed distances corresponding to the edges are positive. Thus, the sum of these positive distances is at least $n+1$, while the sum of the negative distances is at most $n\delta$ in absolute value. Hence $n+1 \leqslant n\delta$, so $\mathbb{R}(\delta)$ can only contain an odd cycle C_{2n+1} if $\delta \geqslant 1+1/n$.

Now for an integer $n \geqslant 2$, define vertices u_i, w_i of $\mathbb{R}(\delta)$ by

$$u_i := i(\delta - 1) \qquad \text{for } 0 \leqslant i < n,$$

$$:= 2 \qquad \text{for } i = n,$$

and $w_i := u_i - 1$ for $0 \leqslant i \leqslant n$. Then u_i is adjacent to w_i in $\mathbb{R}(\delta)$, since $u_i - w_i = 1$; similarly u_0 is adjacent to w_n. Also u_i is adjacent to w_{i-1} for $1 \leqslant i \leqslant n-1$, since in these cases $u_i - w_{i-1} = \delta$. Thus if u_n is adjacent to u_{n-1} we have an odd cycle

$$C_{2n+1} := u_0 w_0 u_1 w_1 \cdots u_{n-2} w_{n-2} u_{n-1} u_n w_n u_0.$$

The condition for u_n and u_{n-1} to be adjacent is

$$1 \leqslant u_n - u_{n-1} = 2 - (n-1)(\delta - 1) \leqslant \delta,$$

that is,

$$1 + \frac{1}{n} \leqslant \delta \leqslant 1 + \frac{1}{n-1}.$$

With appropriate attention to the endpoints of this range, the rest of the theorem now follows. ∎

COROLLARY. *For any positive integer n, the odd cycle C_{2n+1} is the smallest order chromatic subgraph of $\mathbb{R}(\delta)$ exactly when*

$$\delta = 2 \qquad \text{if } n = 1,$$

$$1 + \frac{1}{n} \leqslant \delta < 1 + \frac{1}{n-1} \qquad \text{if } n \geqslant 2.$$

We now specify a class of graphs which will be shown to include a chromatic subgraph of $\mathbb{R}(\delta)$ for each δ in the interval $n-1 < \delta \leqslant n$. For any positive integers m, n let $G(m, n)$ be the graph comprising $m+1$ distinct vertices $u_0, u_1, \ldots, u_m$ and m disjoint subgraphs $H_1, H_2, \ldots, H_m$, each of which is a copy of the complete graph K_n, such that u_0 is adjacent to u_m and each vertex of H_i is adjacent to u_{i-1} and u_i, for $1 \leqslant i \leqslant m$ (Fig. 2). Note that $G(m, 1)$ is the cycle C_{2m+1}, and $G(1, n)$ is the complete graph K_{n+2}.

LEMMA 2. *For any positive integers m, n the graph $G(m, n)$ is colour-critical, with chromatic number $n+2$.*

Proof. Suppose we had a proper colouring of $G(m, n)$ using $n+1$ colours. Without loss of generality the vertex u_0 has colour 0, and the n

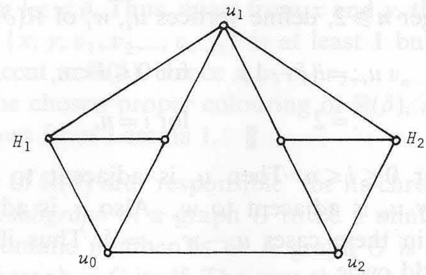

FIG. 2. The graph $G(2, 2)$.

vertices of H_1 have colours 1, 2,..., n. Since u_1 is adjacent to all the vertices of H_1 it follows that u_1 has colour 0. Iterating this argument shows that u_m has colour 0. But this is impossible, since u_m is adjacent to u_0. Hence any proper colouring of $G(m, n)$ requires at least $n+2$ colours. Clearly $n+2$ colours are sufficient, so the chromatic number of $G(m, n)$ is $n+2$. Note that if any edge is removed from $G(m, n)$ the resulting graph has chromatic number $n+1$, so $G(m, n)$ is colour-critical. ∎

THEOREM 4. *For any integers m, $n \geqslant 2$ if $n+1/m \leqslant \delta < n+1/(m-1)$ then $G(m, n)$ is a chromatic subgraph of $\mathbb{R}(\delta)$.*

Proof. We prove the result for the case $n=2$. (The general case is proved similarly but details are more involved.)

Noting that $\delta > 2$, define vertices u_i, v_i, w_i of $\mathbb{R}(\delta)$ by

$$\begin{aligned} u_i &:= i(\delta-2) && \text{for } 0 \leqslant i < m, \\ &:= 1 && \text{for } i = m; \\ v_i &:= u_i - 1 && \text{for } 1 \leqslant i < m, \\ &:= 2 && \text{for } i = m; \\ w_i &:= u_i - 2 && \text{for } 1 \leqslant i < m, \\ &:= 3 && \text{for } i = m. \end{aligned}$$

These vertices are all distinct provided $u_{m-1} < u_m$, $v_{m-1} < u_0$ and $w_{m-1} < v_1$. These conditions amount to $(m-1)(\delta-2) < 1$, which holds since $\delta < 2 + 1/(m-1)$.

Now $|v_i - w_i| = 1$, so v_i is adjacent to w_i in $\mathbb{R}(\delta)$, and these two vertices induce a subgraph H_i isomorphic to K_2, for $1 \leqslant i \leqslant m$. Again $|u_i - v_i| = 1$ and $|u_i + w_i| = 2$, so u_i is adjacent in $\mathbb{R}(\delta)$ to both vertices of H_i, for $1 \leqslant i \leqslant m$. Also $u_m - u_0 = 1$, so u_0 is adjacent to u_m in $\mathbb{R}(\delta)$. Next, for $1 \leqslant i < m$ we have $u_{i-1} - v_i = \delta - 1$ and $u_{i-1} - w_i = \delta$, so u_{i-1} is adjacent in

$\mathbb{R}(\delta)$ to both vertices of H_i, Thus, the vertices u_i, v_i, w_i are vertices of a subgraph $G(m, 2)$ in $\mathbb{R}(\delta)$, provided u_{m-1} is adjacent to both vertices of H_m. This requires $1 \leqslant v_m - u_{m-1} = 2-(m-1)(\delta-2)$ and $\delta \geqslant w_m - u_{m-1} = 3-(m-1)(\delta-2)$, whence $2+1/m \leqslant \delta \leqslant 2+1/(m-1)$. These requirements are satisfied, so $\mathbb{R}(\delta)$ does have $G(m, 2)$ as a subgraph for the specified range of δ. It now follows from Theorem 1 and Lemma 2 that $G(m, 2)$ is a chromatic subgraph of $\mathbb{R}(\delta)$. ▮

COROLLARY. *For any given* $\delta \geqslant 1$, *there are positive integers* m, n *such that* $G(m, n)$ *is a chromatic subgraph of* $\mathbb{R}(\delta)$.

Proof. The case where δ is a positive integer corresponds to $m=1$, and follows from Theorem 2. The case where $1<\delta<2$ corresponds to $n=1$, and follows from Theorem 3. All other cases follow from Theorem 4. ▮

When $\frac{5}{2} \leqslant \delta < 3$, Theorem 4 shows that $G(2, 2)$ is a chromatic subgraph of $\mathbb{R}(\delta)$ (see Fig. 2). By systematically eliminating all other possibilities, it can be shown that any other 4-chromatic subgraph of $\mathbb{R}(\delta)$ in this case either has order greater than 7, or else has order 7 but contains more than 11 edges. In this sense, $G(2, 2)$ is the smallest chromatic subgraph of $\mathbb{R}(\delta)$ when $\frac{5}{2} \leqslant \delta < 3$.

It may be remarked that with due attention to detail, Theorem 4 could be extended to include Theorems 2 and 3. In conjunction with Lemma 1, this would then provide an alternative proof of Theorem 1.

Given any finite graph G, it is clear that $\mathbb{R}(\delta)$ has a subgraph isomorphic to G if δ is sufficiently large. Let d be a positive real number such that $\mathbb{R}(\delta)$ has a subgraph isomorphic to G for all $\delta > d$. It appears that if $d>1$ then there is also a subgraph of $\mathbb{R}(d)$ which is isomorphic to G. However, this is not true if $d=1$; counterexamples are provided by C_4 and $K_{1,3}$, for example.

4. Open Interval of Distances

In this section we consider the case where the distance set D is an *open interval* $(1, \delta)$, for some $\delta \geqslant 1$. We shall use $\mathbb{R}_0(\delta)$ to denote the graph $G(\mathbb{R}, D)$ in this case, and $\chi_0(\delta)$ to denote its chromatic number. (Incidentally, if $\delta=1$ then $D=\varnothing$, so $\mathbb{R}_0(1)$ is an independent set of vertices and $\chi_0(1)=1$.)

It turns out that $\mathbb{R}_0(\delta)$ has the same chromatic number as $\mathbb{R}(\delta)$ when $\delta>1$, so we have

THEOREM 5. *If* $n-1<\delta \leqslant n$ *then* $\chi_0(\delta)=n+1$, *for any integer* $n \geqslant 2$.

Proof. Since $\mathbb{R}_0(\delta)$ is a proper subgraph of $\mathbb{R}(\delta)$, it follows that $\chi_0(\delta) \leqslant \chi(\delta)$ for any δ. Hence if $\delta \leqslant n$, Lemma 1 implies $\chi_0(\delta) \leqslant n+1$. We next show that $\chi_0(\delta) \geqslant n+1$, whence equality follows.

Given any δ such that $n-1 < \delta \leqslant n$, choose δ' so that $n-1 < \delta' < \delta$. By the corollary to Theorem 4, we can find positive integers r, s such that $G(r, s)$ is a chromatic subgraph $\mathbb{R}(\delta')$. Choose any real number c satisfying $1 < c < \delta/\delta'$. Choose any particular subgraph of the form $G(r, s)$ in $\mathbb{R}(\delta')$, and enlarge it by the scale factor c. The new graph is still of the form $G(r, s)$, and each of its edges has length at least c and at most $c\delta'$, that is, the length of each of its edges is strictly between 1 and δ. Hence $\mathbb{R}_0(\delta)$ contains a subgraph $G(r, s)$, so $\chi_0(\delta) \geqslant \chi(\delta')$. But $\chi(\delta') = n+1$, by Theorem 1, so $\chi_0(\delta) \geqslant n+1$. ▮

We can again specify chromatic subgraphs. The method used in proving Theorem 5 adapts to establish the following result.

LEMMA 3. *Let d be a positive real number such that the finite graph G is a subgraph of $\mathbb{R}(\delta)$ if $\delta > d$. Then G is a subgraph of $\mathbb{R}_0(\delta)$ if $\delta > d$.*

Proof. Given any $\delta > d$, choose δ' so that $d < \delta' < \delta$. Then $\mathbb{R}(\delta')$ contains a subgraph G, and we can find a scale factor c, where $1 < c < \delta/\delta'$, so that if this particular subgraph is enlarged by the factor c, the new graph is still isomorphic to G and the length of each of its edges is strictly between 1 and δ, so it is a subgraph of $\mathbb{R}_0(\delta)$, as required. ▮

In fact, the following stronger result holds.

THEOREM 6. *Let d be a positive real number such that $\mathbb{R}(\delta)$ has a subgraph isomorphic to the finite graph G if $\delta > d$, but not if $\delta < d$. Then $\mathbb{R}_0(\delta)$ has a subgraph isomorphic to G if and only if $\delta > d$.*

Proof. Sufficiency follows from Lemma 3. For the necessity of the stated conditions, suppose $\mathbb{R}_0(\delta)$ contains a subgraph G. Let δ' be the greatest length of any edge in this particular subgraph, so $\delta' < \delta$. But this same subgraph is contained in $\mathbb{R}(\delta')$, so $\delta' \geqslant d$. Hence $\delta > d$, and the theorem follows. ▮

COROLLARY. *If $\delta > 1$, no complete graph is a chromatic subgraph of $\mathbb{R}_0(\delta)$.*

Proof. Since Theorem 2 implies that K_{n+1} is a subgraph of $\mathbb{R}(\delta)$ precisely when $\delta \geqslant n$ for any positive integer n, Theorem 6 shows that K_{n+1} is a subgraph of $\mathbb{R}_0(\delta)$ precisely when $\delta > n$. But $\chi_0(\delta) > n+1$ if $\delta > n$, by Theorem 5. The corollary follows. ▮

THEOREM 7. *For any integers $m \geqslant 2, n \geqslant 1$ if $n + 1/m < \delta \leqslant n + 1/(m-1)$ then $G(m, n)$ is a chromatic subgraph of $\mathbb{R}_0(\delta)$.*

Proof. In view of Lemma 3 and Theorem 5, this result is a direct consequence of Theorems 3 and 4. ∎

COROLLARY. *For any $\delta > 1$, there are positive integers m, n such that $G(m, n)$ is a chromatic subgraph of $\mathbb{R}_0(\delta)$.*

We conclude this section by briefly considering $G(\mathbb{R}, D)$ when D is a half-open interval. When $D := [1, \delta)$ for $\delta \geqslant 1$, let $\mathbb{R}_1(\delta)$ denote $G(\mathbb{R}, D)$ and let $\chi_1(\delta)$ be its chromatic number. Similarly $\mathbb{R}_2(\delta)$ and $\chi_2(\delta)$ correspond to the case $D := (1, \delta]$ for $\delta > 1$. As before, $\chi_1(1) = \chi_2(1) = 1$. For $\delta \geqslant 1$ we have $\mathbb{R}_0(\delta) \subset \mathbb{R}_1(\delta) \subset \mathbb{R}(\delta)$ and $\mathbb{R}_0(\delta) \subset \mathbb{R}_2(\delta) \subset \mathbb{R}(\delta)$, whence Theorems 1 and 5 imply that if $n - 1 < \delta \leqslant n$ then $\chi_1(\delta) = n + 1$ and $\chi_2(\delta) = n + 1$, for any integer $n \geqslant 2$.

5. INFINITELY MANY CLOSED INTERVALS

It may at first be thought that an unbounded distance set D would result in the graph $G(\mathbb{R}, D)$ having a transfinite chromatic number. This is not the case, as is shown by the following result, which we shall apply several times.

LEMMA 4. *For any positive integer n, let $D := \bigcup_{k=0}^{\infty} [k(n+1)+1, k(n+1)+n]$. Then the chromatic number of $G(\mathbb{R}, D)$ is at most $n+1$.*

Proof. Let all points of the real line be assigned to colour classes numbered $0, 1, \ldots, n$ such that x has colour i precisely when $\lfloor x \rfloor \equiv i \pmod{n+1}$, as in Fig. 1. We shall show that this is a proper colouring of $G(\mathbb{R}, D)$, whence the lemma.

Let x, y be two reals assigned to colour class i, with $x < y$. Then there are integers r, s such that $r \leqslant s$ and $r(n+1) + i \leqslant x < r(n+1) + i + 1$ and $s(n+1) + i \leqslant y < s(n+1) + i + 1$. If $r = s$ then $0 < y - x < 1$, so x is not adjacent to y in $G(\mathbb{R}, D)$. If $r < s$, let $t := s - r$. Then we have $t(n+1) - 1 < y - x < t(n+1) + 1$, so x and y will be adjacent only if there is some integer $k \geqslant 0$ such that $[k(n+1)+1, k(n+1)+n]$ has a nonempty intersection with $(t(n+1)-1, t(n+1)+1)$. This happens precisely if $k(n+1)+1 < t(n+1)+1$ and $k(n+1)+n > t(n+1)-1$. These inequalities reduce to $k < t$ and $k+1 > t$, which are mutually inconsistent. It follows that the distance between x and y is not in D, so they are not adjacent in $G(\mathbb{R}, D)$, as required. ∎

Note that Lemma 1 is an immediate consequence of Lemma 4, since $[1, n]$ is one of the intervals contained in the distance set in Lemma 4. Another easy application of Lemma 4 is the following result.

THEOREM 8. *If D is any nonempty subset of the odd positive integers, the chromatic number of $G(\mathbb{R}, D)$ is 2.*

Proof. With $n := 1$, the distance set in Lemma 4 is the odd positive integers and the corresponding graph has chromatic number at most 2. This also applies to any subgraph $G(\mathbb{R}, D)$, where D is a nonempty subset of the odd positive integers. Because D is nonempty such a subgraph contains K_2, so its chromatic number is also at least 2. ∎

6. FINITE SET OF DISTANCES

When the distance set D is finite, say $D := \{d_1, d_2, ..., d_m\}$, where $0 < d_1 < d_2 < \cdots < d_m$, we shall write $\mathbb{R}_\alpha(d_1, d_2, ..., d_m)$ to denote $G(\mathbb{R}, D)$, and $\chi_\alpha(d_1, d_2, ..., d_m)$ to denote its chromatic number. Note that the case where $m := 1$ has already been discussed, since $\mathbb{R}_\alpha(\alpha)$ is clearly isomorphic to $\mathbb{R}(1)$.

We shall restrict attention to the cases where each d_i is a positive integer. In such cases, let $d := \gcd\{d_1, d_2, ..., d_m\}$. Then each real number in the interval $[0, d)$ belongs to a different component of $\mathbb{R}_\alpha(d_1, d_2, ..., d_m)$, and clearly every component includes such a real. Moreover, all components of $\mathbb{R}_\alpha(d_1, d_2, ..., d_m)$ are translates of the component containing 0. If D is any subset of the positive integers, it is natural to use $G(\mathbb{Z}, D)$ to denote the graph with the integers as its vertex set, and edges between precisely those pairs of integers with absolute difference in the set D. In particular, for a finite distance set $D := \{d_1, d_2, ..., d_m\}$ we shall denote the graph by $\mathbb{Z}_\alpha(d_1, d_2, ..., d_m)$. Since $\mathbb{Z}_\alpha(d_1, d_2, ..., d_m)$ is a union of components of $\mathbb{R}_\alpha(d_1, d_2, ..., d_m)$, its chromatic number is also $\chi_\alpha(d_1, d_2, ..., d_m)$.

THEOREM 9. *For any positive integer m, the chromatic number of $\mathbb{Z}_\alpha(1, 2, ..., m)$ is $\chi_\alpha(1, 2, ..., m) = m + 1$.*

Proof. By Theorem 1, $\mathbb{R}(m)$ has chromatic number $\chi(m) = m + 1$. Also $\mathbb{Z}_\alpha(1, 2, ..., m)$ is a proper subgraph of $\mathbb{R}(m)$, so $\chi_\alpha(1, 2, ..., m) \leqslant \chi(m)$. On the other hand, the vertices $0, 1, ..., m$ induce a subgraph K_{m+1} in $\mathbb{Z}_\alpha(1, 2, ..., m)$, so $\chi_\alpha(1, 2, ..., m) \geqslant m + 1$ and the theorem follows. ∎

COROLLARY. *For any positive integer m, the complete graph K_{m+1} is a chromatic subgraph of $\mathbb{Z}_\alpha(1, 2, ..., m)$.*

Next we shall consider $\mathbb{Z}_\alpha(d_1, d_2, \ldots, d_m)$ in the particular case $m := 2$. Let r, s be positive integers, with $r < s$. If $\gcd\{r, s\} = d$, then it is clear that $\mathbb{Z}_\alpha(r, s)$ is isomorphic to $\mathbb{Z}_\alpha(r/d, s/d)$. Hence, it suffices to consider the case in which r, s are coprime. If r, s are both odd, Theorem 8 shows that $\chi_\alpha(r, s) = 2$. The remaining case is the subject of the next result.

THEOREM 10. *If r, s are coprime positive integers of opposite parity, with $r < s$, the chromatic number of $\mathbb{Z}_\alpha(r, s)$ is $\chi_\alpha(r, s) = 3$.*

Proof. The graph $\mathbb{Z}_\alpha(r, s)$ contains an odd cycle of order $r + s$, namely $0, r, 2r, \ldots, sr, s(r-1), s(r-2), \ldots, s, 0$. Hence $\chi_\alpha(r, s) \geqslant 3$.

We shall now show that Lemma 4 implies $\chi_\alpha(r, s) \leqslant 3$, whence the theorem. Choose $n := 2$ in Lemma 4, so $D := \bigcup_{k \geqslant 0} [3k+1, 3k+2]$ and the chromatic number of $G(\mathbb{R}, D)$ is at most 3. For any real number $c > 0$, let cD denote the scaled distance set $\{cd : d \in D\}$. Clearly $G(\mathbb{R}, cD)$ is isomorphic to $G(\mathbb{R}, D)$, so has chromatic number at most 3. We shall now show that $\mathbb{Z}_\alpha(r, s)$ is a subgraph of $G(\mathbb{R}, cD)$ for some suitable c, whence $\chi_\alpha(r, s) \leqslant 3$. Indeed, this holds whenever there is a real $c > 0$ and an integer $k \geqslant 0$ such that $c \leqslant r \leqslant 2c$ and $(3k+1)c \leqslant s \leqslant (3k+2)c$. These conditions are equivalent to $r/2 \leqslant c \leqslant r$ and $s/(3k+2) \leqslant c \leqslant s/(3k+1)$. There is a real c satisfying these conditions precisely if $s/(3k+2) \leqslant r$ and $s/(3k+1) \geqslant r/2$, which is equivalent to

$$\frac{1}{3}\left(\frac{s}{r} - 2\right) \leqslant k \leqslant \frac{1}{3}\left(\frac{2s}{r} - 1\right).$$

Let $3a + 2 := s/r$. Then the interval in which k must lie is $a \leqslant k \leqslant 2a + 1$. If $a \geqslant 0$, this interval has length at least 1, and clearly contains at least one positive integer, which may be chosen as k. Since $s/r > 1$ it follows that $a > -\frac{1}{3}$, so if $a < 0$ then $2a + 1 > 0$ and we may choose $k := 0$. In any case, then, there exist suitable numbers c and k so that $\mathbb{Z}_\alpha(r, s)$ is a subgraph of $G(\mathbb{R}, cD)$, as required. ∎

COROLLARY. *If r, s are coprime positive integers of opposite parity, with $r < s$, the odd cycle C_{r+s} is a chromatic subgraph $\mathbb{Z}_\alpha(r, s)$.*

We shall discuss one further class of graphs $G(\mathbb{R}, D)$ with D finite, namely those in which D is an initial segment of the positive integers with one element deleted. Let $D := \{1, 2, \ldots, m\} \setminus \{k\}$, where k, m are positive integers with $k \leqslant m$ and $m \geqslant 2$. Let $\mathbb{R}_\beta(m, k)$ denote $G(\mathbb{R}, D)$ and let $\chi_\beta(m, k)$ denote its chromatic number. Let $\mathbb{Z}_\beta(m, k)$ denote the corresponding subgraph induced by the integers.

When $k := m$, the graph $\mathbb{Z}_\beta(m, m)$ is just $\mathbb{Z}_\alpha(1, 2, \ldots, m-1)$, so we shall restrict attention to the cases where $1 \leqslant k < m$. It is not difficult to obtain the following lemma, and to then use this lemma to derive Theorem 11.

LEMMA 5. *For any positive integer r, the complete graph K_{r+1} is a chromatic subgraph of $\mathbb{Z}_\beta(2r, 1)$, and the graph $G(2, r)$ is a chromatic subgraph of $\mathbb{Z}_\beta(2r+1, 1)$.*

THEOREM 11. *For any integer $m \geqslant 2$, the chromatic number of $\mathbb{Z}_\beta(m, 1)$ is $\chi_\beta(m, 1) = \lfloor \frac{1}{2}(m+3) \rfloor$.*

When $1 < k < m$, the determination of the chromatic number of $\mathbb{Z}_\beta(m, k)$ is more intricate. We shall content ourselves with establishing bounds for $\chi_\beta(m, k)$ in most cases.

First let $k := 2$. Since $\mathbb{Z}_\beta(3, 2)$ is the same as $\mathbb{Z}_\alpha(1, 3)$, its chromatic number is $\chi_\beta(3, 2) = 2$. The next theorem deals with other graphs in this family.

THEOREM 12. *For any positive integer $m \geqslant 4$, the chromatic number of $\mathbb{Z}_\beta(m, 2)$ is $\chi_\beta(m, 2) = \lfloor \frac{1}{2}(m+4) \rfloor$ if $m \not\equiv 3 \pmod 4$, and satisfies $\frac{1}{2}(m+3) \leqslant \chi_\beta(m, 2) \leqslant \frac{1}{2}(m+5)$ if $m \equiv 3 \pmod 4$.*

The above result may be obtained routinely, and its derivation contains the following observation.

COROLLARY. *For any integer $m \geqslant 4$, the graph $\mathbb{Z}_\beta(m, 2)$ contains a subgraph $G(2, \lfloor \frac{1}{2}m \rfloor)$.*

Upper and lower bounds for the general case are provided by the next two theorems. These results may be derived without difficulty.

THEOREM 13. *For any integers k, m with $1 \leqslant k < m$ the chromatic number of $\mathbb{Z}_\beta(m, k)$ satisfies*

$$\chi_\beta(m, k) \leqslant \min\left\{m, \left[\frac{1}{2}\left(\frac{m}{k} + 3\right)\right] k\right\}.$$

THEOREM 14. *For any integers k, m with $3 \leqslant k < m$ the chromatic number of $\mathbb{Z}_\beta(m, k)$ satisfies*

$$\chi_\beta(m, k) \geqslant \max\left\{k, \left[\frac{1}{2}\left(\frac{m}{k-1} + 1\right)\right] t\right\},$$

where $t := 2$ if $k := 3$ and $t := k - 2$ if $k \geqslant 4$.

7. Prime Distances

Let P denote the set of all prime numbers. In this section we shall discuss the chromatic number of $G(\mathbb{R}, D)$ when $D \subseteq P$. Once again, consideration of components shows this to be the same as the chromatic number of $G(\mathbb{Z}, D)$. By analogy with notation used in Section 6, in the case $D := P$ we shall use $\mathbb{Z}_\alpha(P)$ to denote $G(\mathbb{Z}, P)$, and $\chi_\alpha(P)$ to denote its chromatic number.

LEMMA 6. *The chromatic number of $\mathbb{Z}_\alpha(P)$ is $\chi_\alpha(P) = 4$, and the chromatic number of $\mathbb{Z}_\alpha(2, 3, 5)$ is $\chi_\alpha(2, 3, 5) = 4$.*

Proof. Let each integer x be assigned to colour class i precisely when $x \equiv i \pmod 4$, for $0 \leqslant i < 4$. Since integers assigned to the same colour differ by a multiple of 4, they are not adjacent in $\mathbb{Z}_\alpha(P)$, so $\chi_\alpha(P) \leqslant 4$. Because $\mathbb{Z}_\alpha(2, 3, 5)$ is a subgraph of $\mathbb{Z}_\alpha(P)$, we have $\chi_\alpha(2, 3, 5) \leqslant \chi_\alpha(P)$. But $\chi_\alpha(2, 3, 5) \geqslant 4$, since $\mathbb{Z}_\alpha(2, 3, 5)$ contains a subgraph $G(2, 2)$ (see Fig. 2) determined by the vertices $u_0 := 2$, $u_1 := 3$, $u_2 := 4$ and the subgraphs H_1, H_2 with vertex sets $\{0, 5\}$ and $\{1, 6\}$, respectively. The lemma follows. ∎

In view of Lemma 6, we can allocate the subsets D of P to four *classes*, according as $G(\mathbb{Z}, D)$ has chromatic number 1, 2, 3, or 4. Obviously the empty subset is the only member of class 1, and every singleton subset is in class 2. The following theorem addresses the classification of subsets with at least two elements, but does not fully settle this problem.

THEOREM 15. *Let $D \subseteq P$, with $|D| \geqslant 2$. Then D may be classified as follows:*

(a) *D is in class 2 if $2 \notin D$; otherwise D is in class 3 or class 4;*

(b) *if $2 \in D$ and $3 \notin D$, then D is in class 3;*

(c) *if $\{2, 3\} \subseteq D \subseteq \{p \in P : p \equiv \pm 2 \pmod 5\}$ then D is in class 3;*

(d) *if $\{2, 3\} \subseteq D \subseteq \{p \in P : p \equiv \pm 2, \ \pm 3, \ 7 \pmod{14}\}$, then D is in class 3;*

(e) *if $\{2, 3, 5\} \subseteq D$ or $\{2, 3, 11, 13\} \subseteq D$, then D is in class 4.*

Proof. (a) By Theorem 8, if D is a subset of the odd primes then it is in class 2. If D contains 2 and any odd prime p, then $G(\mathbb{Z}, D)$ contains a cycle $v_0 v_1 v_2 \cdots v_p w v_0$, where $v_i := 2i$ for $0 \leqslant i \leqslant p$, and $w := p$. This cycle has order $p + 2$, which is odd, so has chromatic number 3. Hence, with Lemma 6 it follows that D is in class 3 or 4.

(b) If $2 \in D$ and $3 \notin D$, a proper colouring of $G(\mathbb{Z}, D)$ is obtained by assigning the integer x to colour class i precisely when $x \equiv i \pmod 3$, for

$0 \leqslant i < 3$. Integers assigned to the same colour class differ by a multiple of 3, so are not adjacent in $G(\mathbb{Z}, D)$. In view of (a), it follows that D is in class 3.

(c) Assign each integer x to colour class i precisely when $x \equiv 2i$ or $2i+1 \pmod 5$, for $0 \leqslant i \leqslant 1$, and assign x to colour class 2 if $x \equiv 4 \pmod 5$. The difference between any two integers in the same colour class is congruent to 0 or $\pm 1 \pmod 5$, so no such pair is adjacent in $G(\mathbb{Z}, D)$ if $D \subseteq \{p \in P: p \equiv \pm 2 \pmod 5\}$. With (a), if $2 \in D$ it follows that D is in class 3.

(d) Assign each integer x to colour class i, where $i := 0$ when $x \equiv 0, 1, 5, 6$, or $10 \pmod{14}$, $i := 1$ when $x \equiv 2, 3, 7, 11$, or $12 \pmod{14}$, and $i := 2$ when $x \equiv 4, 8, 9$, or $13 \pmod{14}$. The difference between any two integers in the same colour class is congruent to $0, \pm 1, \pm 4, \pm 5$, or $\pm 6 \pmod{14}$, so no such pair is adjacent in $G(\mathbb{Z}, D)$ if $D \subseteq \{p \in P: p \equiv \pm 2, \pm 3$, or $7 \pmod{14}\}$. With (a), if $2 \in D$ it follows that D is in class 3.

(e) If $\{2, 3, 5\} \subseteq D$, then D is in class 4, by Lemma 6. If $\{2, 3, 11, 13\} \subseteq D$, then D can be seen to be in class 4 by the following argument. In view of (a), suppose we can find a proper colouring of the integers in $G(\mathbb{Z}, D)$ using only three colours. Not every pair of consecutive integers is assigned to the same colour class, so without loss of generality we may suppose the integers 0 and 1 are assigned to colour classes 0 and 1, respectively. For convenience, we shall write $x \to i$, for $0 \leqslant i \leqslant 2$, if the integer x is assigned to colour class i. Then consecutively we deduce $0 \to 0$, $1 \to 1$, $3 \to 2$, $14 \to 0$, $12 \to 2$, $-1 \to 0$, $-2 \to 2$, $11 \to 1$, $9 \to 0$, $6 \to 1$, $-10 \to 0$, $-13 \to 1$, $-11 \to 2$, $-8 \to 1$, $-5 \to 0$, $8 \to 2$, $-3 \to 1$. But 10 is adjacent to -3, -1, and 12, which have been assigned to three different colour classes, so there is no proper colouring of $G(\mathbb{Z}, D)$ with only three colours. ∎

Although many sets $D \subseteq P$ are not classified by Theorem 15, among others it does classify all subsets of $\{2, 3, 5, 7, 11, 13, 17\}$. Clearly it also classifies all sets D with 2 elements, and many with 3. The next result classifies all remaining 3-sets.

THEOREM 16. *For any prime $p > 5$, the set $\{2, 3, p\}$ is in class* 3.

Proof. In view of Theorem 15, it suffices to demonstrate a proper colouring of $\mathbb{Z}_\alpha(2, 3, p)$ using three colours. First suppose $p := 6k+1$, for some positive integer k. Assign each integer x to colour class i, where $0 \leqslant i \leqslant 2$, as follows. Let $x \equiv a \pmod{6k+4}$ with $-4 \leqslant a < 6k$. If $a \geqslant 0$ then $x \to i$ when $a \equiv 2i$ or $2i+1 \pmod 6$. Also $x \to 0$ if $a = -4$; $x \to 1$ if $a = -3$ or -2; $x \to 2$ if $a = -1$. No two integers in the same colour class differ by $+2$ or $\pm 3 \pmod{6k+4}$, so this is a proper colouring of $\mathbb{Z}_\alpha(2, 3, 6k+1)$.

Now suppose $p := 6k-1$, for some integer $k \geqslant 2$. This time, for any integer x let $x \equiv a \pmod{6k+2}$, with $-14 \leqslant a < 6k-12$. If $a \geqslant 0$ then $x \to i$ when $a \equiv 2i$ or $2i+1 \pmod 6$, with $0 \leqslant i \leqslant 2$. As in the proof of Theorem 15(d), when $a < 0$ we assign $x \to 0$ if $a \in \{-14, -13, -9, -8, -4\}$, $x \to 1$ if $a \in \{-12, -11, -7, -3, -2\}$, and $x \to 2$ if $a \in \{-10, -6, -5, -1\}$. No two integers in the same colour class differ by ± 2 or $\pm 3 \pmod{6k+2}$, so this is a proper colouring of $\mathbb{Z}_\alpha(2, 3, 6k-1)$, with $k \geqslant 2$. This covers all cases. ∎

We conclude this section with a theorem in the spirit of some of our earlier results on chromatic subgraphs.

THEOREM 17. *Let $D \subseteq P$. If K_n is a complete subgraph of largest order in $G(\mathbb{Z}, D)$ then $n \leqslant 4$, and*

(a) *$n \geqslant 3$ precisely when $\{2, p, q\} \subseteq D$, where p, q are twin primes;*

(b) *$n = 4$ precisely when $\{2, 3, 5, 7\} \subseteq D$.*

Proof. Because K_n has chromatic number n, Lemma 5 implies that $n \leqslant 4$ if K_n is a subgraph of $G(\mathbb{Z}, D)$, with $D \subseteq P$.

Suppose $G(\mathbb{Z}, D)$ contains a subgraph K_3. Such a subgraph cannot have two edges of even length, for then the longest edge would be even and at least 4, so could not be prime. Thus at least two edges of any K_3 must have odd length, so the third edge must have even length which, being prime, must be 2. The lengths of the two odd edges must therefore be twin primes. Conversely, if D contains 2 and a pair of twin primes, it is obvious that $G(\mathbb{Z}, D)$ contains K_3.

Next suppose $G(\mathbb{Z}, D)$ contains a complete subgraph K_4. Then there is a subgraph K_3 which, in view of what we have just proved, can be taken to have vertices $0, p, q$, where p, q are odd primes with $q = p+2$. If the fourth vertex of K_4 is $v < q$, the subgraph K_3 with vertices $0, q, v$ must have one edge of length 2, and $v \neq p$, so it follows that $v = \pm 2$. If $v = -2$, the adjacency of v and q requires that $p+4$ also be prime; if $v = 2$, the adjacency of v and p requires that $p-2$ also be prime. So in either case D must contain 2 and two intersecting pairs of twin primes, whence $\{2, 3, 5, 7\} \subseteq D$. Conversely, if $\{2, 3, 5, 7\} \subseteq D$ then $G(\mathbb{Z}, D)$ contains a subgraph K_4 with vertices 0, 2, 5, 7. ∎

8. CONCLUDING REMARKS

By analogy with the notation used in this paper, it is natural to use $G(\mathbb{R}^2, \{1\})$ to denote the graph on all points of the euclidean plane, with any two points adjacent precisely if their distance apart is 1. It is instructive to note that the proof by Moser and Moser [2] that the chromatic number of $G(\mathbb{R}^2, \{1\})$ is at least 4 is in fact a proof that $G(2, 2)$ is a subgraph. The identification of chromatic subgraphs relevant to $G(\mathbb{R}, D)$ for various distance sets D, additional to those discussed in the present paper, would not only be of intrinsic interest, but might also be relevant to graphs of the type $G(\mathbb{R}^2, D)$.

The colourings we have used in proving our results either explicitly use monochromatic intervals or (when the distance set is a subset of the positive integers) easily yield colourings of the real line with monochromatic intervals. What differences result if it is required that every open interval contains at least two colours or, more generally, at least k colours, for some prescribed k?

A number of problems raised in the paper were not solved. These are indicated in the relevant sections. We hope to discuss, in a sequel to this paper, a class of problems not raised here: what happens when the distance set is not bounded away from 0?

References

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