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CHROMATIC NUMBER OF FINITE AND INFINITE GRAPHS AND HYPERGRAPHS

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We wrote many papers on these subjects, some in collaboration with Galvin, Rado, Shelah and Szemerĕdi, and posed many problems some of which turned out to be undecidable. In this survey we state some old and new solved and unsolved problems.

Nous avons écrit beaucoup d'articles, certains en collaboration avec Galvin, Rado, Shelah et Szemerĕdi, au sujet du nombre chromatique des graphes et des hypergraphes, finis ou infinis. Nous avons posé bien des problèmes, dont certains se sont avérés indécidables. Dans cette brève synthèse nous présentons quelques problèmes, anciens ou récents, résolus ou encore en suspens.

First we state a few simple finite problems.

Chromatic graphs attracted the attention of mathematicians first of all because of the four colour problem, but it was soon realized that interesting and difficult questions can be proved on chromatic graphs which have nothing to do with the four colour problem (or Appel and Haken theorem).

Tutte (and independently Ungar, Zykov and Mycielski) proved that for every k there are k-chromatic graphs without a triangle.

First of all, we study sparse graphs of high chromatic number. Erdös proved that for every k and l there is a k-chromatic graph of girth $\geqslant l$. More precisely, there is a graph on n vertices of girth $\geqslant l$, the largest independent set of which is $< n^{1-c/l}$ for a certain absolute constant c (independent of l and n). Thus the chromatic number is $> n^{c/l}$. For related problems and results see [1–3].

Now we state a few extremal problems.

Denote by $c(3, n)$ the smallest integer c for which there is a graph on c vertices of chromatic number n which has no triangle. $c(3, 3) = 5$ is trivial and it is known that $c(3, 4) = 11$; as far as we know $c(3, 5)$ has not yet been determined or at least is not known to us. It would not be hard to determine $c(3, 5)$ but the determination of $c(3, n)$ for larger values of n will be very difficult. It is not even trivial to prove that $c(3, n)/n \to \infty$. This was first shown by Erdös who in fact proved $c(3, n) > n^{1+\alpha}$ and also $r(3, n) > n^{1+\alpha}$, $r(3, n)$ is the smallest integer for which every graph of $r(3, n)$ vertices either has a triangle or an independent set of n vertices. The sharpest results known at present are

$$c_1 \frac{n^2}{(\log n)^2} < r(3, n) < c_2 \frac{n^2}{\log n}. \tag{1}$$

The lower bound is due to Erdös, the upper bound is due to Ajtai, Komlós and

$(i<2, n<\omega)$ and $(0, n)$ $(1, m)$ are joined by an edge if $n<m$. Thomassen independently gave a different proof which does not use the half-graph theorem.

More recently Hajnal and Komjath improved the half-graph theorem by showing that if the chromatic number is uncountable then the graph contains a half-graph $H(\aleph_0, \aleph_0)$ and one extra vertex which is joined to all the vertices of H having infinite degree in H. This result is surprisingly sharp in the sense that, if C.H. holds, then there need not be two such vertices.

Taylor asked the following fundamental question: let $\mathcal{G}$ be any graph of chromatic number $\aleph_1$. Is it true that, for every cardinal $m \geqslant \aleph_1$, there is a graph $\mathcal{H}$ of chromatic number m all of whose finite subgraphs are contained in $\mathcal{G}$? This beautiful problem (which in a special case of a more general question in logic) is very far from being solved. As far as we know our paper with Shelah is the only one on this subject, and there we state several other interesting problems and conjectures.

Beyond the result of Hajnal and Komjath nothing is known about the countable graphs that must be contained in every graph of chromatic number $\aleph_1$. It should be possible to obtain further results of this kind since every such graph must, of course, contain a countable graph of chromatic number $\aleph_0$.

We conjectured that if $\mathcal{G}$ has chromatic number $\aleph_0$, then $\sum 1/n_i = \infty$, where $n_1 < n_2 < \cdots$ are the lengths of the circuits in $\mathcal{G}$. We conjectured further that this is true under the weaker assumption that the edge density is infinite (i.e., if e_n is the largest number of edges among any n points, then $e_n/n \to \infty$).

Komlos and Szemerĕdi settled this by showing that if a graph G on n points has edge density k, then $\sum 1/n_i > c \log k$. Possibly $\sum 1/n_i$ is minimal for the complete bipartite graph. If the chromatic number is k, perhaps $\sum 1/n_i$ is minimal for the complete graph K_k. As far as we know the following deeper problems are still open: if $\mathcal{G}$ has chromatic number $\aleph_0$ is it true that the sum $\sum\{1/n_i : n_i \text{ odd}\}$ is also infinite? Also, for graphs of chromatic number k does this sum tend to infinity with k?

Here is another question posed by Mihoc and Erdös at a party at Nesetril's during the meeting in Prague. Burr and Erdös had earlier conjectured and Bollobas proved that, if a graph $\mathcal{G}$ on n vertices has no circuit whose length is in the sequence $(a, a+d, a+2d, \ldots)$, and is not all terms of this progression are odd, then $\mathcal{G}$ has fewer than $C_{a,d}n$ edges, and hence has chromatic number $<1+C_{a,d}$. The new question is this. What properties on the sequence of integers $q_1 < q_2 < q_3 < \cdots$ will ensure that if a graph has no circuit of length q_i $(i = 1, 2, 3, \ldots)$, then the chromatic number on the edge density of $\mathcal{G}$ is finite? Probably the answers for these two questions will not be the same. It is clear that if the q_i increase sufficiently fast the chromatic number can be $\aleph_0$, but what happens for example if $q_i = 2^i$? Is it true that every graph of infinite chromatic number contains a circuit C_{2^i} for infinitely many i? There has not yet been sufficient time to judge yet whether there is a fruitful question. Similarly, of course, one can ask if graphs with infinite edge density must contain C_{2^i} for infinitely many i.

An old problem of ours which seems to have been neglected is the following. Is it true that every graph $\mathcal{G}$ of chromatic number $\aleph_1$ contains a subgraph $\mathcal{G}'$ of infinite connectivity and chromatic number $\aleph_1$? Komjath claims to have shown that there are subgraphs with connectivity n (for every $n < \aleph_0$). We constructed a graph of power $\aleph_1$ and chromatic number $\aleph_1$ such that every subset of $\aleph_1$ vertices, there is one which is jointed to only countably many of the others.

Another interesting problem we asked with Szemerĕdi is the following. For any graph $\mathcal{G}$ of chromatic number $\aleph_1$ there is, of course, a finite subgraph of chromatic number n for every finite n. Denote by $h_{\mathcal{G}}(n)$ the smallest integer h such that $\mathcal{G}$ contains a finite subgraph of size h and chromatic number n. Is there a universal function f such that, for every $\mathcal{G}$, $h_{\mathcal{G}}(n) < f(n)$ for all sufficiently large n? We proved that $h_{\mathcal{G}}$ increases faster than the r-times iterated exponential. From our earlier remarks it is easy to see that there is no such universal function for graphs of chromatic number $\aleph_0$.

Another old question of ours is the following. Is it true that for every k, n there is $f_k(n)$ such that every graph of chromatic number $f_k(n)$ contains a subgraph of chromatic number n and girth at least k? Nesetril and Rödl proved this for $k=4$. Even if the existence of such a function f_k is known, the problem still remains to determine the slowest possible growth for f_k. The infinite version of this problem is unsolved.

Galvin asked for the Darboux property of chromatic numbers. More precisely: Is it true that if $\mathcal{G}$ has chromatic number m then it contains a subgraph of chromatic number n for $\aleph_0 < n < m$? Subgraph here can be interpreted either as induced subgraph or not. In the induced case Galvin has certain negative results under the assumption that $2^{\aleph_0} = \aleph_2$. Perhaps the Darboux property should be investigated for other graph theoretical properties.

In a recent paper with Szemerĕdi we prove that for every infinite cardinal m there is an m-chromatic graph each finite subgraph of which is almost bipartite in the following sense: for every finite subgraph on k vertices, εk of these can be omitted so that the resulting subgraph is bipartite. However, we do not know if there is such a graph of cardinality m. In fact, we do not know if there is a graph of cardinality and chromatic number m such that every k-element set of vertices contains an independent set of size ck.

We also investigated corresponding problem on edges. Denote by $f_{\mathcal{G}}(k)$ the smallest integer such that every induced subgraph of $\mathcal{G}$ with k vertices can be made bipartite by the omission of at most $f_{\mathcal{G}}(k)$ edges. We proved that $f_{\mathcal{G}}(k) < 2k^{\frac{3}{2}}$ and $f_{\mathcal{G}}(k) \neq \mathcal{O}(k)$. For a finite graph $\mathcal{G}$, Rödl proved $f_{\mathcal{G}}(k) < ck$ [6].

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