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SOME RESULTS IN COMBINATORIAL NUMBER THEORY

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We should like to state here some new results in combinatorial number theory.

1. In [3] p.50 the following question was asked: "Let $A=\{a_1<a_2<\dots\}$ and $B=\{b_1<b_2<\dots\}$ be sequences of integers satisfying $A(x) > \varepsilon x^{1/2}$, $B(x) > \varepsilon x^{1/2}$ for some $\varepsilon>0$. Is it true that

$$a_i - a_j = b_k - b_t \tag{1}$$

has infinitely many solutions?" ($A(x)$ and $B(x)$ are the number of elements of A and B up to x, resp.)

R. Freud observed that the answer was negative: we write the numbers in binary scale, and select for A those ones which contain only even powers of two, and for

B those which contain only odd powers of two. Then (1) is possible only in the trivial case, and

$$m = \liminf_{x\to\infty} \frac{\min\{A(x),\ B(x)\}}{\sqrt{x}} = \frac{1}{\sqrt{2}}\ .$$

Later P. Erdős and R. Freud investigated general properties of sequences A and B for which (1) has only trivial solutions. We state here some of these results:

1.1. $m > 2^{-1/4} - \varepsilon$ is attainable.

1.2. The largest possible value of $S = \limsup_{x\to\infty} A(x)B(x)/x$ is 2. More precisely, if $\limsup_{n} H(n) = \infty$ then $A(n)B(n) \geq 2n - H(n)$ is attainable for infinitely many n by suitable A and B, but $A(n)B(n) - 2n \to -\infty$ for any A and B.

1.3. $s = \liminf_{x\to\infty} A(x)B(x)/x$ is at most $14/9$, more precisely

$$(5/2)s + 2S \leq 7.$$

Also $s + (3/2)S \leq 4$, which shows that $S=2$ implies $s\leq 1$. It is not yet known whether $s>1$ is possible at all.

1.4. If $m>0$, then neither $A(x)/\sqrt{x}$ nor $B(x)/\sqrt{x}$ can tend to a limit.

Several further theorems are proved on the behaviour of $A(x)B(x)/x$, $A(x)/\sqrt{x}$ and $B(x)/\sqrt{x}$. The results with detailed proofs will appear in [1], and another forthcoming paper will deal with related problems.

2. Now we consider permutations of integers. In the finite case let $a_1,a_2,\dots,a_n$ be a permutation of the integers $1,2,\dots,n$, and in the infinite case let $a_1,a_2,\dots,a_i,\dots$ be a permutation of all positive integers.

P.Erdős, R.Freud and N.Hegyvári investigated several estimations concerning the values of $[a_i,a_{i+1}]$ and (a_i,a_{i+1}).

2.1. In the finite case

$$\min \max_{1\le i\le n-1} [a_i,a_{i+1}] = (1+o(1))\,\frac{n^2}{4\log n}$$

where the minimum is to be taken over all permutations $a_1,\dots,a_n$.

This result can somewhat be improved if we omit $o(n)$ numbers from $1,2,\dots,n$; then:

2.2. $\min \max[a_i, a_{i+1}] < n^{2-\varepsilon(n)}$ is attainable for any $\varepsilon(n) \to 0$; but is impossible for a fix $\varepsilon > 0$.

2.3. In the infinite case there exists a permutation satisfying

$$[a_i, a_{i+1}] < ie^{c\sqrt{\log i}\,\log\log i} \quad \text{for all } i.$$

In the opposite direction we can show only that

$$\lim_i \sup \frac{[a_i, a_{i+1}]}{i} \geq \frac{1}{1-\log 2}$$

must hold for any permutation.

2.4. Concerning the greatest common divisors, we can construct an infinite permutation with

$$(a_i, a_{i+1}) > i/2 \quad \text{for all } i.$$

In the opposite direction we can show only that

$$\lim_i \inf \frac{(a_i, a_{i+1})}{i} \leq \frac{61}{90}$$

is valid for any permutation.

These results with detailed proofs will appear in [2].

3. Concerning the permutations now we consider the values of a_i+a_{i+1}. Odlyzko has constructed an infinite permutation where a_i+a_{i+1} is always a prime (see [3, p.94]). R.Freud and N.Hegyvári investigated several other sets with can contain the values of a_i+a_{i+1} for a suitable infinite permutation. E.g. the set of the squares has this property, and a pair of residue classes, k and $k+s \pmod m$ has this property if and only if either $(s,m)=1$, or $(s,m)=2$ and k is odd (one residue class alone is clearly "bad").

In [4] the density of the sums of subsequent elements is investigated. It is shown, that the identical permutation is in some sense best possible, but this is not the case from several other points of view. As a special case also an answer is given to a question of N.Hegyvári, which asked for the smallest possible value of the maximum of the sums a_i+a_{i+1}, $1\le i\le n-1$ concerning all such permutations of $1,2,\dots,n$ where the sums a_i+a_{i+1} are distinct.

We quote here the results for the infinite case. For an infinite permutation put

$$T = \{t \mid t = a_i + a_{i+1} \text{ is solvable for some } i\},$$

and

$$R = \limsup_{x\to\infty} \frac{T(x)}{x}, \quad r = \liminf_{x\to\infty} \frac{T(x)}{x}.$$

3.1. The largest possible value of R is $2/3$. More precisely, if $\limsup_n H(n) = \infty$, then

$$T(n) > (2/3)n - H(n)$$

can be attained for infinitely many n by a suitable permutation, but for any permutation

$$T(n) - (2/3)n \to -\infty.$$

Moreover $R=2/3$ implies $r=0$.

3.2. The largest possible value of r is $1/2$. More precisely $T(n) \equiv [(n-1)/2]$ can be attained (e.g. by the identical permutation), but $T(n) < [n/2]$ must hold for infinitely many n for any permutation. Moreover, $r=1/2$ implies $R=1/2$.

3.3. In general

$$r \leq \frac{R(2-3R)}{1-2R^2}.$$

The attractive conjectures

$$3R + r \leq 2, \quad R + r \leq 1 \quad \text{and} \quad Rr \leq 1/4$$

satisfied by the extremal permutations are false:

3.4. For suitable permutations we have

$$3R + r \geq 2{,}184, \quad R + r \geq 1{,}1018 \quad \text{and}$$

$$Rr \geq 0{,}257.$$

(On the other hand 3.3. assures that

$$3R + r \leq 2{,}236, \quad R + r \leq 1{,}042 \quad \text{and}$$

$$Rr \leq 0{,}270$$

hold for any permutation.)

In [4] the finite case and the generalization for more than two terms is considered as well.

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