COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
34. TOPICS IN CLASSICAL NUMBER THEORY
BUDAPEST (HUNGARY), 1981.

ON THE STATISTICAL THEORY OF PARTITIONS

P. ERDŐS and M. SZALAY

1.

In what follows we are dealing with some statistical properties of partitions resp. unequal partitions of positive integers.

Let Π be a generic "unrestricted" partition of the positive integer n, i.e.,

$$(1.1)\qquad \Pi : \left\{ \begin{array}{l} \lambda_1+\lambda_2+\ldots+\lambda_m=n, \quad \lambda_1\geq\lambda_2\geq\ldots\geq\lambda_m (\geq 1) \\ \lambda_j \text{'s integers}, \quad m=m(\Pi) \end{array} \right\}.$$

As to the number $p(n)$ of unrestricted partitions of n, according to the theorem of G.H. HARDY and S. RAMANUJAN from 1918 (see [3]), we have

(1.2) $$p(n) = (1+O(\frac{1}{\sqrt{n}}))\ \frac{1}{4n\sqrt{3}}\ \exp(\frac{2\pi}{\sqrt{6}}\ \sqrt{n}).^{*}$$

Suppose that $\omega(n)\nearrow\infty$ (arbitrarily slowly). Then, the theorem of P. ERDŐS and J. LEHNER from 1941 (see [1]) states that, for *almost all* partitions of n, i.e., with the exception of $o(p(n))$ Π 's at most, we have

(1.3) $$\lambda_1 = \frac{\sqrt{6}}{2\pi}\ \sqrt{n}\ \log n + O(\sqrt{n}\ \omega(n))$$

and

(1.4) $$m = \frac{\sqrt{6}}{2\pi}\ \sqrt{n}\ \log n + O(\sqrt{n}\ \omega(n)).$$

Further, the relation

$$\lambda_1 \le \frac{\sqrt{6}}{2\pi}\ \sqrt{n}\ \log n + \frac{\sqrt{6}}{\pi}\ \sqrt{n}\cdot c \quad (c\ \text{ real constant})$$

holds for

$$(\exp(-\ \frac{\sqrt{6}}{\pi}\ e^{-c}) + o(1))\ p(n)$$

partitions of n.

* Here the O-sign and later the o-sign refer to $n\to\infty$.

Here,

$$\lambda_1(\Pi) = \max_{\mu} \{\lambda_\mu | \lambda_\mu \in \Pi\},$$

i.e.,

$$\lambda_1' = \max_{\mu} \lambda_\mu .$$

In this paper we consider the problem of the asymptotic behaviour of the maximum with multiplicities, i.e., of

$$(1.5) \qquad \max_{\mu}\{\lambda_\mu \text{ mult}(\lambda_\mu) | \lambda_\mu \in \Pi\}$$

for *almost all* partitions of n.

This maximum is trivially

$$\geq \lambda_1 \cdot 1 \geq \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n - \sqrt{n} \ \omega(n),$$

but one can expect that

$$\max_{\mu}\{\lambda_\mu \text{ mult}(\lambda_\mu)\} \geq \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n + f(n)\sqrt{n}$$

(with a suitable function $f(n)>\omega(n)$) for almost all Π's.

It is trivial that

$$\text{mult}(\lambda_1) = 1$$

for almost all Π 's, moreover, since 1 appears as summand $[\sqrt{n}/\omega(n)]$ -times at least, we have

$$\lambda_1 - \lambda_2 \geq [\sqrt{n}/\omega(n)]$$

for almost all Π 's. Similarly, removing 1's and adding to the maximal summand and conversely, it is easy to see that the inequality

$$(1.6) \qquad \lambda_1 - \lambda_2 \geq \frac{\sqrt{6}}{\pi} \sqrt{n} \cdot c \quad (c \text{ positive constant})$$

holds for

$$(1.7) \qquad (e^{-c} + o(1))\, p(n)$$

partitions of n.

Adding and removing summands, for *one* k, we obtain that $\text{mult}(\lambda_\mu)=k$ implies that

$$\lambda_\mu \text{ mult}(\lambda_\mu) = \lambda_\mu k \leq \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n + \sqrt{n}\, \omega(n)$$

for *almost all* partitions of n. However, for *many* k's, the mentioned transformation is not injective (it may happen that $k'\lambda'_{\mu'} = k''\lambda''_{\mu''}$). At the same time we see that in order to get a significant increase we have to consider *many* k's. Indeed, by deeper arguments we shall prove the following

THEOREM 1. *Let us suppose that* $\omega(n)\nearrow\infty$ *(arbitrarily slowly). Then, for almost all partitions of* n, *we have*

$$(1.8)\qquad \max_{\mu}\{\lambda_\mu \operatorname{mult}(\lambda_\mu)\} =$$

$$= \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n + \frac{\sqrt{6}}{\pi}\sqrt{n}\operatorname{logloglog} n + o(\sqrt{n}\,\omega(n)).$$

Moreover, the number of partitions of n *with the property*

$$(1.9)\qquad \max_{\mu}\{\lambda_\mu \operatorname{mult}(\lambda_\mu)\} \le$$

$$\le \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n + \frac{\sqrt{6}}{\pi}\sqrt{n}\operatorname{logloglog} n + \frac{\sqrt{6}}{\pi}\sqrt{n}\cdot c$$

(c *real constant)*

is

(1.10) $\quad (\exp(-\frac{\sqrt{6}}{\pi} e^{-c}) + o(1))\, p(n).$

2.

At the Colloquium on Number Theory in Debrecen (1974) M. SZALAY (see [4]) gave an application of Erdős-Lehner's theorem to the dimensions of the complex irreducible representations of S_n, the symmetric group on n letters. According to a theorem of G. Frobenius and I. Schur, there is an appropriate one-to-one correspondence between the unrestricted partitions of n and the pairwise non-equivalent irreducible representations (over the complex field) of S_n : $\Pi \leftrightarrow \Gamma_\Pi$ such that

$$\dim \Gamma_\Pi = n! \frac{\prod_{1\le\mu<\nu\le m} (\lambda_\mu - \lambda_\nu + \nu - \mu)}{\prod_{\mu=1}^{m} (\lambda_\mu + m - \mu)!} .$$

M.P. Schützenberger called the attention to the interest of the question what can be said on the distribution of these dimensions. M. SZALAY [4] showed that almost all dimensions are of the form

(2.1) $\quad \exp\{\frac{1}{2} n \log n - O(n \log\log n)\}.$

P. ERDŐS gave a simpler proof (considering the contribution of the primes from the interval $[\lambda_1+m, n]$) and remarked (see [7], pp. 154-155) that this *cannot* be improved to

$$(2.2) \qquad \exp\{g(n) + o(n^{1/2} \log^{-1} n)\}$$

for any function $g(n)$.

By means of the statistical investigation of the λ_μ's, the estimation (2.1) was improved to

$$(2.3) \qquad \exp\{\tfrac{1}{2} n \log n - (\tfrac{1}{2}+A)n + o(n^{7/8} \log^4 n)\}$$

by M. SZALAY and P. TURÁN (see [5], [6], [7]). Here A is a *positive* constant defined by complicated integrals. This estimation required relatively precise asymptotics for the λ_μ's. E.g., with the restriction

$$(2.4) \qquad \log^6 n \le \mu \le \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n - 5\sqrt{n} \log\log n$$

the relation

$$(2.5) \qquad \lambda_\mu = \left(1+o\left(\frac{1}{\log n}\right)\right) \frac{\sqrt{6}}{\pi} \sqrt{n} \log \frac{1}{1-\exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right)}$$

holds *uniformly* with the exception of $O(p(n)n^{-1})$ partitions of n.

We can mention also a simple consequence of this estimation, connected with our first problem. Let μ_0 be the minimal index with the property

$$\text{mult}(\lambda_{\mu_0}) \geq 2 .$$

It is easy to see that (for $\omega(n)\nearrow\infty$)

$$\mu_0 \leq n^{1/4}\,\omega(n)$$

in almost all partitions of n. Hence

$$\lambda_{\mu_0} \geq \lambda_{[n^{1/4}\omega(n)]} \geq \frac{\sqrt{6}}{4\pi}\sqrt{n}\log n - \sqrt{n}\,\omega(n)$$

and, consequently,

$$\text{mult}(\lambda_{\mu_0}) = 2.$$

($\text{mult}(\lambda_{\mu_0})\geq 3$ would imply a contradictory upper estimation similarly to the case $k=3$.)

As another application of (2.4)-(2.5), we proved that almost all partitions of n represent all integers from $[1,n]$ as subsums (see P. ERDŐS-M. SZALAY [2]).

In these cases the summands were in decreasing order. We can investigate the increasing order, i.e.,

$$\lambda'_\mu = \lambda_{m+1-\mu} .$$

For $\omega(n)\nearrow\infty$, the number of 1's is between $\sqrt{n}/\omega(n)$ and $\sqrt{n}\,\omega(n)$ (in almost all partitions of n). Hence,

$$\lambda'_\mu = 1$$

for

$$1 \le \mu \le \sqrt{n}/\omega(n).$$

Apart from this triviality, we can obtain good estimations only for "large" μ's. However, the above multiplicity problem does not appear in *unequal* partitions:

$$(2.6) \qquad \Pi^* : \left\{ \begin{array}{l} \alpha_1+\ldots+\alpha_m=n, \quad \alpha_1>\alpha_2>\ldots>\alpha_m (\ge 1) \\ \alpha_j \text{'s integers}, \quad m=m(\Pi^*) \end{array} \right\}$$

and

(2.7) $\alpha'_{\mu} = \alpha_{m+1-\mu}$.

Their number $q(n)$ is

$$\sim \frac{1}{4n^{3/4}\ 3^{1/4}} \exp\left(\frac{\pi}{\sqrt{3}} \sqrt{n}\right)$$

(see G.H. HARDY-S. RAMANUJAN [3]).

As to the decreasing order, following [5], [6] and [7], one can obtain analogous estimations, roughly, instead of

$$\frac{\sqrt{6}}{\pi} \sqrt{n} \log \frac{1}{1-\exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right)}$$

with

$$\frac{2\sqrt{3}}{\pi} \sqrt{n} \log \frac{1}{\exp\left(\frac{\pi\mu}{2\sqrt{3n}}\right)-1}$$

(the ranges change).

However, the increasing order is more interesting. We shall prove the following

THEOREM 2. *For* $\omega_1(n)\nearrow\infty$, $\omega_2(n)\nearrow\infty$ *and*

$$(2.8) \qquad \omega_1(n) \log n \le \mu \le \frac{\sqrt{n}}{\omega_2(n)} ,$$

we have the uniform relation

$$(2.9) \qquad \alpha'_\mu = (1+o(\frac{\mu}{\sqrt{n}} + \sqrt{\frac{\log n}{\mu}}))\, 2\mu$$

apart from $O(q(n)n^{-1})$ *unequal partitions of* n.

Here the upper bound for μ is *necessary* to $\alpha'_\mu \sim 2\mu$ (cf. (4.20)) but the lower bound in (2.8) is too large and we can prove also

THEOREM 3. *For*

$$(2.10) \qquad (2^8 \le) k_0 \le \mu \le n^{1/6} ,$$

we have the uniform estimation

$$(2.11) \qquad |\alpha'_\mu - 2\mu| \le \mu \sqrt{\frac{40 \log k_0}{k_0}}$$

with the exception of

$$(2.12) \qquad O(q(n)k_0^{-3/2})$$

unequal partitions of n.

Theorem 3 and the proof of Theorem 2 (cf. (4.22)-(4.23)) yield the following

COROLLARY. *For arbitrary* $\eta>0$, *there exist* n_0 and $\varepsilon>0$ *such that, for* $n>n_0$ *with the restriction*

$$(2.13) \qquad \frac{1}{\varepsilon} \le \mu \le \varepsilon\sqrt{n} \ ,$$

the estimation

$$(2.14) \qquad |\alpha'_\mu - 2\mu| \le \eta\mu$$

holds uniformly with the exception of

$$(2.15) \qquad \eta q(n)$$

unequal partitions of n *at most.*

3. PROOF OF THEOREM 1.

We shall use, for

$$(3.1) \qquad \text{Re } z > 0,$$

the function

(3.2) $$f(z) \overset{\text{def}}{=} \prod_{\nu=1}^{\infty} \frac{1}{1-\exp(-\nu z)} = 1 + \sum_{n=1}^{\infty} p(n)\exp(-nz)$$

and the well-known formula

(3.3) $$f(z) = \exp\left(\frac{\pi^2}{6z} + \frac{1}{2} \log \frac{z}{2\pi} + o(1)\right) \quad \text{for} \quad z \to 0$$

in all angles

(3.4) $$|\arc z| \leq \varkappa < \pi/2$$

(log means the principal logarithm).

For arbitrary positive integers n and k, let $p(n,k)$ denote the number of partitions of n containing ν as summand $[k/\nu]$-times *at most* for *every* integer ν of the interval $[1,k]$. Let us observe that the relation

(3.5) $$1 + \sum_{n=1}^{\infty} p(n,k)w^n =$$

$$= \left\{ \prod_{\nu=1}^{k} (1-w^{([k/\nu]+1)\nu}) \right\} \prod_{\nu=1}^{\infty} \frac{1}{1-w^{\nu}}$$

holds for $|w|<1$. Cauchy's formula gives the representation

(3.6) $$p(n,k) =$$

$$= \frac{1}{2\pi i} \int_{|w|=\rho} w^{-n-1} \{ \prod_{\nu=1}^{k} (1-w^{([k/\nu]+1)\nu}) \} \prod_{\nu=1}^{\infty} \frac{1}{1-w^{\nu}} \, dw$$

for $0<\rho<1$. Let us define $g_k(z)$ by

(3.7) $$g_k(z) =$$

$$= \{ \prod_{\nu=1}^{k} (1-\exp(-\nu z([k/\nu]+1))) \} f(z)$$

for

(3.8) $$x = \mathrm{Re}\ z > 0.$$

Then we have

(3.9) $$p(n,k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_k(x+iy)\exp(nx+iny)dy$$

for $x>0$.

Let c_0 be a sufficiently large constant and ε fixed with $0<\varepsilon<10^{-2}$, further,

(3.10) $$\frac{\sqrt{6}}{2\pi} \sqrt{n} \log n \le k \le \frac{\sqrt{6}}{\pi} \sqrt{n} \log n.$$

We choose

$$(3.11) \qquad x = x_0 = \frac{\pi}{\sqrt{6}} n^{-1/2}, \quad y_1 = n^{-3/4+\varepsilon/3}, \quad y_2 = c_0 x_0$$

and investigate (3.9) as

$$p(n,k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_k(x_0+iy)\exp(nx_0+iny)dy =$$

$$= \frac{1}{2\pi} \{ \int_{-\pi}^{-y_2} + \int_{-y_2}^{-y_1} + \int_{-y_1}^{y_1} + \int_{y_1}^{y_2} + \int_{y_2}^{\pi} \} .$$

(We use some ideas of G.A. Freiman's $p(n)$ -estimation.)

For

$$(3.12) \qquad |y| \le y_2 \qquad (\text{and } n\to\infty),$$

we can apply (3.3)-(3.4) and get

$$f(x_0+iy) =$$

$$= \exp(\frac{\pi^2}{6(x_0+iy)} + \frac{1}{2} \log \frac{x_0+iy}{2\pi} + o(1)),$$

further, under the restriction (3.10),

$$|\exp(-\nu(x_0+iy)([k/\nu]+1))| =$$

$$= \exp(-\nu x_0([k/\nu]+1)) \leq \exp(-kx_0) \leq n^{-1/2},$$

$$\prod_{\nu=1}^{k} (1-\exp(-\nu(x_0+iy)([k/\nu]+1))) =$$

$$= \exp(-\sum_{\nu=1}^{k} \log \frac{1}{1-\exp(-\nu(x_0+iy)([k/\nu]+1))}) =$$

$$= \exp(-\sum_{\nu=1}^{k} \{\exp(-\nu(x_0+iy)([k/\nu]+1)) +$$

$$+ O(\exp(-2kx_0))\}) =$$

$$= \exp(-\sum_{\nu=1}^{k} \exp(-\nu(x_0+iy)([k/\nu]+1)) + o(1)).$$

Consequently, the relation

(3.13) $$g_k(x_0+iy) =$$

$$= \exp(\frac{\pi^2}{6(x_0+iy)} + \frac{1}{2}\log\frac{x_0+iy}{2\pi} -$$

$$- \sum_{\nu=1}^{k} \exp(-\nu(x_0+iy)([k/\nu]+1)) + o(1))$$

holds under the restrictions (3.10)-(3.12).

First,

$$\frac{1}{2\pi}\int_{-y_1}^{y_1} g_k(x_0+iy)\exp(nx_0+iny)dy =$$

$$= \frac{1}{2\pi}\int_{-y_1}^{y_1}\exp\{nx_0+iny + \frac{\pi^2}{6x_0}(1-i\frac{y}{x_0} - (\frac{y}{x_0})^2 +$$

$$+ o((\frac{y_1}{x_0})^3)) + o(1) + \frac{1}{2}\log\frac{x_0}{2\pi} + o(\frac{y_1}{x_0}) -$$

$$- \sum_{\nu=1}^{k}(1+o(ky_1))\exp(-\nu x_0([k/\nu]+1))\}dy =$$

$$= \frac{1}{2\pi}\int_{-y_1}^{y_1}\exp\{\frac{2\pi}{\sqrt{6}}\sqrt{n} + \frac{1}{2}\log\frac{x_0}{2\pi} - \frac{n}{x_0}y^2 + o(1) -$$

$$- \sum_{\nu=1}^{k}\exp(-\nu x_0([k/\nu]+1)) + o(k^2 y_1\exp(-kx_0))\}dy =$$

$$= \frac{1+o(1)}{2\pi}\sqrt{\frac{x_0}{2\pi}}\exp(\frac{2\pi}{\sqrt{6}}\sqrt{n} - \sum_{\nu=1}^{k}\exp(-\nu x_0([k/\nu]+1))) \times$$

$$\times \int_{-y_1}^{y_1}\exp(-\frac{n}{x_0}y^2)dy =$$

$$= \frac{1+o(1)}{4n\sqrt{3}\sqrt{\pi}} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n} - \sum_{\nu=1}^{k} \exp(-\nu x_0([k/\nu]+1))\right) \times$$

$$\times \left\{ \int_{-\infty}^{+\infty} \exp(-u^2)\,du + o(1) \right\} =$$

$$= \frac{1+o(1)}{4n\sqrt{3}} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n} - \sum_{\nu=1}^{k} \exp(-\nu x_0([k/\nu]+1))\right) =$$

$$= (1+o(1))p(n)\exp\left(-\sum_{\nu=1}^{k} \exp(-\nu x_0([k/\nu]+1))\right).$$

Next,

$$\frac{1}{2\pi} \left| \int_{-y_2}^{-y_1} + \int_{y_1}^{y_2} \right| \le$$

$$\le \frac{1}{\pi} \int_{y_1}^{y_2} \exp\left(nx_0 + \frac{\pi^2 x_0}{6(x_0^2+y^2)} + \frac{1}{2}\log\frac{x_0}{2\pi} + o(1) +\right.$$

$$\left. + \sum_{\nu=1}^{k} \exp(-\nu x_0([k/\nu]+1))\right)dy \le$$

$$\le \exp\left(nx_0 + \frac{\pi^2 x_0}{6(x_0^2+y_1^2)} + \frac{1}{2}\log\frac{x_0}{2\pi} +\right.$$

$$+ o(1) + k \exp(-kx_0)) =$$

$$= \exp(nx_0 + \frac{\pi^2}{6x_0}(1 - \frac{y_1^2}{x_0^2} + o(\frac{y_1^4}{x_0^4})) + o(\log n)) =$$

$$= \exp(\frac{2\pi}{\sqrt{6}} \sqrt{n} - \frac{\sqrt{6}}{\pi} n^{3/2} y_1^2 + o(\log n)) =$$

$$= p(n)\exp(-\sum_{\nu=1}^{k} \exp(-\nu x_0([k/\nu]+1)) -$$

$$- \frac{\sqrt{6}}{\pi} n^{3/2} y_1^2 + o(\log n)) =$$

$$= o(1)p(n)\exp(-\sum_{\nu=1}^{k} \exp(-\nu x_0([k/\nu]+1))).$$

Finally, we have to estimate the expression

$$\frac{1}{2\pi} \left| \int_{-\pi}^{-y_2} + \int_{y_2}^{\pi} \right|.$$

For

$$(3.14) \qquad y_2 \le |y| \le \pi,$$

we get

$$|f(x_0+iy)| =$$

$$= \exp(\mathrm{Re} \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\infty} \frac{1}{\mu} \exp(-\nu\mu(x_0+iy))) \leq$$

$$\leq \exp(\sum_{\mu=1}^{\infty} \frac{1}{\mu} |\exp(\mu(x_0+iy))-1|^{-1}) =$$

$$= \exp(\sum_{\mu=1}^{\infty} \frac{1}{\mu} ((\exp(\mu x_0)-1)^2 \quad +$$

$$+ 4\exp(\mu x_0)\sin^2 \frac{\mu y}{2})^{-1/2}) \leq$$

$$\leq \exp((2\sin \frac{|y|}{2})^{-1} + \sum_{\mu=2}^{\infty} \frac{1}{\mu^2 x_0}) \leq$$

$$\leq \exp(\frac{1}{x_0}(\frac{\pi^2}{6} - 1 + \frac{\pi}{2c_0}))$$

and

$$|\prod_{\nu=1}^{k} (1-\exp(-\nu(x_0+iy)([k/\nu]+1)))| \leq$$

$$\leq \prod_{\nu=1}^{k} (1+\exp(-\nu x_0([k/\nu]+1))) \leq$$

$$\leq (1+\exp(-kx_0))^k \leq \exp(k\exp(-kx_0)).$$

Therefore,

$$\frac{1}{2\pi} \left| \int_{-\pi}^{-y_2} + \int_{y_2}^{\pi} \right| \leq$$

$$\leq \frac{1}{\pi} \int_{y_2}^{\pi} \exp(nx_0 + \frac{1}{x_0}(\frac{\pi^2}{6} - 1 + \frac{\pi}{2c_0}) + .$$

$$+ k\exp(-kx_0))dy \leq$$

$$\leq \exp(\frac{2\pi}{\sqrt{6}}\sqrt{n} - (1 - \frac{\pi}{2c_0})\frac{1}{x_0} + O(\log n)) =$$

$$= p(n)\exp(-\sum_{\nu=1}^{k}\exp(-\nu x_0([k/\nu]+1)) -$$

$$- (1 - \frac{\pi}{2c_0})\frac{\sqrt{6}}{\pi}\sqrt{n} + O(\log n)) =$$

$$= o(1)p(n)\exp(-\sum_{\nu=1}^{k}\exp(-\nu x_0([k/\nu]+1))).$$

Thus we have proved the relation

(3.15) $p(n,k) =$

$$= (1+o(1))p(n)\exp\Big(-\sum_{\nu=1}^{k} \exp(-\nu x_0([k/\nu]+1))\Big)$$

for

(3.16) $$x_0 = \frac{\pi}{\sqrt{6}}\, n^{-1/2},$$

$$\frac{\sqrt{6}}{2\pi}\sqrt{n}\,\log n \le k \le \frac{\sqrt{6}}{\pi}\sqrt{n}\,\log n.$$

Supposing the inequalities

(3.17) $$\frac{\sqrt{6}}{2\pi}\sqrt{n}\,\log n + \frac{\sqrt{6}}{2\pi}\sqrt{n}\,\operatorname{logloglog} n \le k \le$$

$$\le \frac{\sqrt{6}}{\pi}\sqrt{n}\,\log n,$$

we obtain that

$$\sum_{\nu=1}^{k} \exp(-\nu x_0([k/\nu]+1)) =$$

$$= \sum_{\nu=1}^{[k/([\log n]+1)]} \exp(-\nu x_0([k/\nu]+1)) +$$

$$+\sum_{s=1}^{[\log n]}\sum_{\nu=[k/(s+1)]+1}^{[k/s]} \exp(-\nu x_0([k/\nu]+1)) =$$

$$= O(k \log^{-1} n) \exp(-kx_0) +$$

$$+\sum_{s=1}^{[\log n]}\sum_{\nu=[k/(s+1)]+1}^{[k/s]} \exp(-\nu x_0(s+1)) =$$

$$= O((\log\log n)^{-1/2}) +$$

$$+\sum_{s=1}^{[\log n]}\left(\frac{\exp(-x_0(s+1)([k/(s+1)]+1))}{1-\exp(-x_0(s+1))} -\right.$$

$$\left.-\frac{\exp(-x_0(s+1)([k/s]+1))}{1-\exp(-x_0(s+1))}\right)=$$

$$= o(1) + \sum_{s=1}^{[\log n]}\left(\frac{\exp(-x_0(s+1)[k/(s+1)])}{\exp(x_0(s+1))-1} -\right.$$

$$\left.-\frac{\exp(-x_0(s+1)[k/s])}{\exp(x_0(s+1))-1}\right)=$$

$$= o(1) + \sum_{s=1}^{[\log n]}\left(\frac{\exp(-kx_0+O(x_0(s+1)))}{\exp(x_0(s+1))-1} -\right.$$

$$- \frac{\exp(-kx_0(1+s^{-1})+o(x_0(s+1)))}{\exp(x_0(s+1))-1}) =$$

$$= o(1) + \exp(-kx_0) \sum_{s=1}^{[\log n]} \frac{1-\exp(-kx_0 s^{-1})}{\exp(x_0(s+1))-1} +$$

$$+ \exp(-kx_0) \sum_{s=1}^{[\log n]} \frac{o(x_0(s+1))}{\exp(x_0(s+1))-1} = \exp(-kx_0) \times$$

$$\times \sum_{s=1}^{[\log n]} (1-\exp(-kx_0 s^{-1}))\{\frac{1}{x_0(s+1)} + O(1)\} + o(1) =$$

$$= x_0^{-1} \exp(-kx_0) \sum_{s=1}^{[\log n]} \frac{1-\exp(-kx_0 s^{-1})}{s+1} + o(1).$$

We apply the trivial inequalities

$$0 < \frac{\exp(-kx_0 s^{-1})}{s+1} \leq \frac{\exp(-(2s)^{-1}\log n)}{s+1} \leq$$

$$\leq \begin{cases} \frac{1}{\log n} & \text{if} \quad 1 \leq s \leq \frac{\log n}{2\log\log n} \\ \frac{1}{s+1} & \text{if} \quad \frac{\log n}{2\log\log n} < s \leq \log n. \end{cases}$$

Thus,

$$0 < \sum_{s=1}^{[\log n]} \frac{1}{s+1} \exp(-kx_0 s^{-1}) \le$$

$$\le O(\log n) \frac{1}{\log n} + \log(2\log\log n) + O(1) =$$

$$= O(\log\log\log n).$$

Hence, by (3.17),

$$\sum_{\nu=1}^{k} \exp(-\nu x_0([k/\nu]+1)) = x_0^{-1} \exp(-kx_0) \times$$

$$\times \{\log\log n + O(\log\log\log n)\} + o(1) =$$

$$= \frac{\sqrt{6}}{\pi} \sqrt{n} \log\log n \exp(-kx_0) + o(1).$$

Therefore, we have

(3.18) $p(n,k) =$

$$= (\exp(- \frac{\sqrt{6}}{\pi} \sqrt{n} \log\log n \exp(- \frac{\pi k}{\sqrt{6n}})) +$$

$$+ o(1))\, p(n)$$

for

$$(3.19)\qquad \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n + \frac{\sqrt{6}}{2\pi}\sqrt{n}\,\mathrm{logloglog}\, n \le k \le$$

$$\le \frac{\sqrt{6}}{\pi}\sqrt{n}\log n.$$

Taking into account ERDŐS-LEHNER's theorem, (3.19) implies that, for almost all Π's, every summand is $\le k$. Hence, Theorem 1 follows from (3.18)-(3.19).

4. PROOF OF THEOREM 2.

For

$$(4.1)\qquad \mathrm{Re}\, z > 0,$$

we have

$$(4.2)\qquad 1 + \sum_{n=1}^{\infty} q(n)\exp(-nz) = \prod_{\nu=1}^{\infty} (1+\exp(-\nu z)).$$

(3.2), (3.3) and (3.4) give that

$$(4.3)\qquad \prod_{\nu=1}^{\infty} (1+\exp(-\nu z)) = \frac{f(z)}{f(2z)} =$$

$$= \exp\left(\frac{\pi^2}{12z} - \frac{1}{2}\log 2 + o(1)\right) \quad \text{for} \quad z \to 0$$

in all angles

$$(4.4) \qquad |\arc z| \le \varkappa < \pi/2.$$

For arbitrary nonnegative integers n, k and $\Lambda \ge 1$, let $g(n,k,\Lambda)$ denote the number of *unequal* partitions of n with *exactly* k summands $\le \Lambda$; here by definition

$$(4.5) \qquad g(0,0,\Lambda) = 1; \quad g(0,k,\Lambda) = 0 \quad \text{for} \quad k \ge 1.$$

Let us observe that the relation

$$(4.6) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g(n,k,\Lambda)\exp(-nx-ky) =$$

$$= \prod_{\ell=1}^{\Lambda} (1+\exp(-\ell x-y)) \prod_{\nu=\Lambda+1}^{\infty} (1+\exp(-\nu x))$$

holds for real y and positive x. We choose

$$(4.7) \qquad x = x_0 = \frac{\pi}{2\sqrt{3}}\, n^{-1/2}.$$

First, for

$$(4.8) \qquad y > 0, \quad n \to \infty, \quad K \geq 0,$$

we obtain that

$$\exp(-nx_0 - Ky) \sum_{k=0}^{K} g(n,k,\Lambda) \leq$$

$$\leq \sum_{k=0}^{K} g(n,k,\Lambda) \exp(-nx_0 - ky) \leq$$

$$\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g(n,k,\Lambda) \exp(-nx_0 - ky) =$$

$$= \prod_{\nu=1}^{\infty} (1+\exp(-\nu x_0)) \prod_{\ell=1}^{\Lambda} \frac{1+\exp(-\ell x_0 - y)}{1+\exp(-\ell x_0)} ,$$

i.e.,

$$\sum_{k=0}^{K} g(n,k,\Lambda) \leq$$

$$\leq \exp(nx_0 + Ky) \prod_{\nu=1}^{\infty} (1+\exp(-\nu x_0)) \times$$

$$\times \prod_{\ell=1}^{\Pi} \left(1 - \frac{1-\exp(-y)}{\exp(\ell x_0)+1}\right) \leq$$

$$\leq \exp(nx_0+Ky+\frac{\pi^2}{12x_0}+o(1)-$$

$$-(1-\exp(-y))\sum_{\ell=1}^{\Lambda}\frac{1}{\exp(\ell x_0)+1}) = o(n^{3/4}q(n))\times$$

$$\times\exp(Ky-y\exp(-y)\sum_{\ell=1}^{\Lambda}\frac{1}{\exp(\ell x_0)+1}).$$

For

$$K \leq e^{-y}\sum_{\ell=1}^{\Lambda}\frac{1}{\exp(\ell x_0)+1} - 3\,\frac{\log n}{y}, \tag{4.9}$$

we get

$$\sum_{k=0}^{K} g(n,k,\Lambda) = o(q(n)n^{-2}). \tag{4.10}$$

Next, for

$$y < 0, \quad n \to \infty, \quad L \geq 0, \tag{4.11}$$

we obtain that

$$\exp(-nx_0-Ly)\sum_{k=L}^{\infty} q(n,k,\Lambda) \leq$$

$$\leq \sum_{k=L}^{\infty} g(n,k,\Lambda)\exp(-nx_0-ky) \leq$$

$$\leq \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g(n,k,\Lambda)\exp(-nx_0-ky) =$$

$$= \prod_{\nu=1}^{\infty}(1+\exp(-\nu x_0))\prod_{\ell=1}^{\Lambda}\frac{1+\exp(-\ell x_0+|y|)}{1+\exp(-\ell x_0)} ,$$

i.e.,

$$\sum_{k=L}^{\infty} g(n,k,\Lambda) \leq$$

$$\leq \exp(nx_0-L|y|)\prod_{\nu=1}^{\infty}(1+\exp(-\nu x_0)) \times$$

$$\times \prod_{\ell=1}^{\Lambda}\left(1+\frac{\exp(|y|)-1}{\exp(\ell x_0)+1}\right) \leq$$

$$\leq \exp\Big(nx_0-L|y| + \frac{\pi^2}{12x_0} + o(1) +$$

$$+ (\exp(|y|)-1)\sum_{\ell=1}^{\Lambda}\frac{1}{\exp(\ell x_0)+1}\Big) =$$

$$= O(n^{3/4}q(n)) \times$$

$$\times \exp\Big(-L|y|+|y|\exp(|y|)\sum_{\ell=1}^{\Lambda}\frac{1}{\exp(\ell x_0)+1}\Big) .$$

For

$$(4.12) \qquad L \geq e^{|y|} \sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} + 3\,\frac{\log n}{|y|}\,,$$

we get

$$(4.13) \qquad \sum_{k=L}^{\infty} g(n,k,\Lambda) = O(q(n)n^{-2}).$$

For an unequal partition Π^* (cf. (2.6)-(2.7)), let $s(n,\Pi^*,\Lambda)$ stand for the number of summands (in Π^*) not exceeding Λ. (4.8)-(4.10) and (4.11)-(4.13) yield that the inequalities

$$(4.14) \qquad s(n,\Pi^*,\Lambda) \geq e^{-t} \sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} - 3\,\frac{\log n}{t}\,,$$

$$(4.15) \qquad s(n,\Pi^*,\Lambda) \leq e^{t} \sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} + 3\,\frac{\log n}{t}$$

hold for arbitrary $t>0$ with the exception of $O(q(n)n^{-2})$ unequal partitions of n at most.

We suppose that

$$(4.16) \qquad \Lambda \geq \log n$$

and choose

(4.17) $$t = t_0 = \sqrt{\frac{6\log n}{\Lambda}} .$$

Then, from (4.14) and (4.15),

$$s(n,\Pi^*,\Lambda) - \sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} \geq$$

$$\geq -(1-\exp(-t_0)) \sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} - 3\,\frac{\log n}{t_0} \geq$$

$$\geq -t_0\Lambda 2^{-1} - 3t_0^{-1}\log n =$$

$$= -\sqrt{6\Lambda}\sqrt{\log n} ,$$

resp.,

$$s(n,\Pi^*,\Lambda) - \sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} \leq$$

$$\leq (\exp(t_0)-1) \sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} + 3\,\frac{\log n}{t_0} \leq$$

$$\leq \exp(t_0)t_0\Lambda 2^{-1} + 3t_0^{-1} \log n \leq$$

$$\leq \exp(\sqrt{6})\sqrt{6\Lambda \log n} .$$

Thus, the estimation

(4.18) $\quad S(n,\Pi^*,\Lambda) =$

$$= \sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} + O(\sqrt{\Lambda \log n})$$

holds for $\Lambda \geq \log n$ with the exception of $O(q(n)n^{-2})$ unequal partitions of n at most. The exceptional sets can vary with Λ but the relation (4.18) holds *uniformly* for $\log n \leq \Lambda \leq n$ with the exception of

(4.19) $\quad O(q(n)n^{-1})$

unequal partitions of n at most.

We have

$$\sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} = \int_0^{\Lambda} \frac{1}{\exp(\vartheta x_0)+1}\, d\vartheta + O(1) =$$

$$= \left[- \frac{1}{x_0} \log(1+\exp(-\vartheta x_0))\right]_0^{\Lambda} + O(1) =$$

$$= \frac{1}{x_0} \log \frac{2}{1+\exp(-\Lambda x_0)} + O(1).$$

Consequently, the uniform relation

(4.20) $s(n, \Pi^*, \Lambda) =$

$$= \frac{2\sqrt{3}}{\pi} \sqrt{n} \log \frac{2}{1+\exp(-\frac{\pi\Lambda}{2\sqrt{3n}})} + O(\sqrt{\Lambda \log n})$$

holds for all but $O(q(n)n^{-1})$ unequal partitions of n with the restriction

(4.21) $\log n \le \Lambda \le n.$

(For $\Lambda = c\sqrt{n}$, cf. P. ERDŐS-J. LEHNER [1], Theorem 3.1.)

Since $s(n, \Pi^*, \alpha'_\mu) = \mu$, (4.20) shows the *necessity* of the *upper* bound in (2.8) to $\alpha'_\mu \sim 2\mu$.

If Λ is restricted by

(4.22) $\log n \le \Lambda \le (2x_0)^{-1} = \frac{\sqrt{3}}{\pi} \sqrt{n},$

we obtain that

$$\frac{\Lambda}{2} \ge \sum_{\ell=1}^{\Lambda} \frac{1}{\exp(\ell x_0)+1} = \sum_{\ell=1}^{\Lambda} \frac{\exp(-\ell x_0)}{1+\exp(-\ell x_0)} \ge$$

$$\ge \frac{1}{2} \sum_{\ell=1}^{\Lambda} \exp(-\ell x_0) \ge \frac{1}{2} \sum_{\ell=1}^{\Lambda} (1-\ell x_0) \ge$$

$$\geq \frac{\Lambda}{2} - \frac{\Lambda^2 x_0}{2} (\geq \frac{\Lambda}{4}) \ .$$

Therefore, the uniform relation

$$(4.23) \qquad s(n,\Pi^*,\Lambda) =$$

$$= \frac{\Lambda}{2} + O(\Lambda^2 x_0) + O(\sqrt{\Lambda \log n})$$

holds for all but $O(q(n)n^{-1})$ unequal partitions of n with the restriction (4.22). Let

$$(4.24) \qquad \omega_1(n)\log n \leq \mu \leq \frac{\sqrt{n}}{\omega_2(n)} \quad \text{and} \quad n>n_0 \ .$$

Then, by (4.22)-(4.23),

$$s(n,\Pi^*,3\mu) = \frac{3}{2}\mu + o(\mu) > \mu \ ,$$

$$s(n,\Pi^*,\frac{3}{2}\mu) = \frac{3}{4}\mu + o(\mu) < \mu \ .$$

Hence,

$$\frac{3}{2}\mu < \alpha'_\mu < 3\mu \ .$$

Thus α'_μ satisfies (4.22)(and $\alpha'_\mu = o(\mu)$). By (4.23),

$$\mu = S(n, \Pi^*, \alpha'_\mu) =$$

$$= \frac{\alpha'_\mu}{2} + O(\alpha'^2_\mu x_0) + O(\sqrt{\alpha'_\mu \log n}) =$$

$$= \frac{\alpha'_\mu}{2} + O(\mu^2 x_0) + O(\sqrt{\mu \log n})$$

which implies that

$$\alpha'_\mu = 2\mu + O(\mu^2 x_0) + O(\sqrt{\mu \log n}) =$$

$$= (1 + O(\frac{\mu}{\sqrt{n}}) + O(\sqrt{\frac{\log n}{\mu}}))2\mu .$$

5. PROOF OF THEOREM 3.

For arbitrary positive integers n and k, let $q_k(n)$ denote the number of *unequal* partitions (of n) *every* summand of which is *greater* than k. We shall need the following

LEMMA. *For*

(5.1) $k = k(n)$, $1 \le k \le n^{1/5}$ *and* $n \to \infty$, *the relation*

$$(5.2)\qquad q_k(n) = (1+o(1))\,\frac{q(n)}{2^k}$$

holds.

PROOF. Let us observe that the relation

$$(5.3)\qquad 1 + \sum_{n=1}^{\infty} q_k(n)w^n = \prod_{\nu=k+1}^{\infty} (1+w^\nu)$$

holds for $|w|<1$. Cauchy's formula gives the representation

$$(5.4)\qquad q_k(n) = \frac{1}{2\pi i} \int_{|w|=\rho} w^{-n-1} \prod_{\nu=k+1}^{\infty} (1+w^\nu)\,dw$$

for $0<\rho<1$. Let us define $h_k(z)$ by

$$(5.5)\qquad h_k(z) = \prod_{\nu=k+1}^{\infty} (1+\exp(-\nu z))$$

for

$$(5.6)\qquad x = \operatorname{Re} z > 0.$$

Then we have

$$(5.7)\qquad q_k(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_k(x+iy)\exp(nx+iny)dy$$

for $x>0$.

Let c_0 be a sufficiently large constant and ε fixed with $0<\varepsilon<10^{-2}$, further,

$$(5.8)\qquad 1\le k\le n^{1/5}.$$

We choose

$$(5.9)\qquad x = x_0 = \frac{\pi}{2\sqrt{3}}\, n^{-1/2},\quad y_1 = n^{-3/4+\varepsilon/3},$$

$$y_2 = c_0 x_0$$

and investigate (5.7) as

$$q_k(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_k(x_0+iy)\exp(nx_0+iny)dy =$$

$$= \frac{1}{2\pi}\left\{\int_{-\pi}^{-y_2} + \int_{-y_2}^{-y_1} + \int_{-y_1}^{y_1} + \int_{y_1}^{y_2} + \int_{y_2}^{\pi}\right\}.$$

For

$$(5.10) \qquad |y| \le y_2 \qquad (\text{and} \quad n\to\infty),$$

we can apply (4.3)-(4.4) and get

$$\prod_{\nu=1}^{\infty} (1+\exp(-\nu(x_0+iy))) =$$

$$= \exp\left(\frac{\pi^2}{12(x_0+iy)} - \frac{1}{2}\log 2 + o(1)\right),$$

further

$$\prod_{\nu=1}^{k} (1+\exp(-\nu(x_0+iy)))^{-1} =$$

$$= \exp\left(-\sum_{\nu=1}^{k} \log(1+\exp(-\nu(x_0+iy)))\right) =$$

$$= \exp\left(-k \log 2 + \sum_{\nu=1}^{k} \log \frac{1}{1-2^{-1}(1-\exp(-\nu(x_0+iy)))}\right).$$

Here, by (5.8) and (5.10),

$$\left| \sum_{\nu=1}^{k} \log \frac{1}{1-2^{-1}(1-\exp(-\nu(x_0+iy)))} \right| \le$$

$$\le \sum_{\nu=1}^{k} \log \frac{1}{1-2^{-1}|1-\exp(-\nu(x_0+iy))|} \le$$

$$\le \sum_{\nu=1}^{k} \frac{2^{-1}|1-\exp(-\nu(x_0+iy))|}{1-2^{-1}|1-\exp(-\nu(x_0+iy))|} =$$

$$= \sum_{\nu=1}^{k} O(\nu n^{-1/2}) = O(k^2 n^{-1/2}) =$$

$$= O(n^{-1/10}) = o(1).$$

Consequently, the relation

$$(5.11) \qquad h_k(x_0+iy) =$$

$$= \exp(-k \log 2 + \frac{\pi^2}{12(x_0+iy)} - \frac{1}{2} \log 2 + o(1))$$

holds under the restrictions (5.8) and (5.10).

First,

$$\frac{1}{2\pi} \int_{-y_1}^{y_1} h_k(x_0+iy)\exp(nx_0+iny)dy =$$

$$= \frac{1}{2\pi} \int_{-y_1}^{y_1} \exp\{-k \log 2+\frac{\pi^2}{12x_0}(1-i\frac{y}{x_0}-(\frac{y}{x_0})^2+O((\frac{y_1}{x_0})^3)) +$$

$$+nx_0+iny-\frac{1}{2}\log 2+o(1)\}dy =$$

$$= \frac{1}{2\pi} \int_{-y_1}^{y_1} \exp\{-k \log 2 + \frac{\pi}{\sqrt{3}} \sqrt{n} -$$

$$- \frac{2\sqrt{3}}{\pi} n^{3/2} y^2 + O(n^{-1/4+\varepsilon}) - \frac{1}{2}\log 2 + o(1)\} dy =$$

$$= \frac{1+o(1)}{2\pi\sqrt{2}} 2^{-k} \exp(\frac{\pi}{\sqrt{3}} \sqrt{n}) \times$$

$$\times \int_{-y_1}^{y_1} \exp(- \frac{2\sqrt{3}}{\pi} n^{3/2} y^2) dy =$$

$$= \frac{1+o(1)}{2\pi\sqrt{2}} 2^{-k} \exp(\frac{\pi}{\sqrt{3}} \sqrt{n}) \sqrt{\frac{\pi}{2}} 3^{-1/4} n^{-3/4} \times$$

$$\times \{\int_{-\infty}^{+\infty} \exp(-u^2) du + o(1)\} =$$

$$= \frac{1+o(1)}{4n^{3/4} 3^{1/4}} 2^{-k} \exp(\frac{\pi}{\sqrt{3}} \sqrt{n}) =$$

$$= (1+o(1)) 2^{-k} q(n).$$

Next,

$$\frac{1}{2\pi}\left| \int_{-y_2}^{-y_1} + \int_{y_1}^{y_2} \right| \le$$

$$\le \frac{1}{\pi} \int_{y_1}^{y_2} \exp(-k \log 2 + \frac{\pi^2 x_0}{12(x_0^2+y^2)} + nx_0 + O(1))dy =$$

$$= o(1)2^{-k} \exp(\frac{\pi^2 x_0}{12(x_0^2+y_1^2)} + nx_0) =$$

$$= o(1)2^{-k} \exp(\frac{\pi^2}{12x_0}(1 - \frac{y_1^2}{x_0^2} + O(\frac{y_1^4}{x_0^4}))+nx_0) =$$

$$= o(1)2^{-k} \exp(\frac{\pi}{\sqrt{3}} \sqrt{n} - \frac{2\sqrt{3}}{\pi} n^{2\varepsilon/3}) =$$

$$= o(1)2^{-k} q(n).$$

Finally, we have to estimate the expression

$$\frac{1}{2\pi} \left| \int_{-\pi}^{-y_2} + \int_{y_2}^{\pi} \right|.$$

For

(5.12) $\quad y_2 \le |y| \le \pi$,

we get (with $z=x_0+iy$)

$$\exp(\mathrm{Re}\sum_{\nu=k+1}^{\infty}\log(1+\exp(-\nu z))) =$$

$$= \exp(\mathrm{Re}\sum_{\nu=k+1}^{\infty}\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu-1}}{\mu}\exp(-\nu\mu z)) =$$

$$= \exp(\mathrm{Re}\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu-1}\exp(-k\mu z)}{\mu(\exp(\mu z)-1)}) \leq$$

$$\leq \exp(\sum_{\mu=1}^{\infty}\mu^{-1}|\exp(\mu z)-1|^{-1}) =$$

$$= \exp(\sum_{\mu=1}^{\infty}\mu^{-1}((\exp(\mu x_0)-1)^2 +$$

$$+ 4\exp(\mu x_0)\sin^2\frac{\mu y}{2})^{-1/2}) \leq$$

$$\leq \exp((2\sin\frac{|y|}{2})^{-1} + \sum_{\mu=2}^{\infty}\frac{1}{\mu^2 x_0}) \leq$$

$$\leq \exp(\frac{1}{x_0}(\frac{\pi^2}{6} - 1 + \frac{\pi}{2c_0})).$$

Therefore,

$$\frac{1}{2\pi}\left| \int_{-\pi}^{-y_2} + \int_{y_2}^{\pi} \right| \le$$

$$\le \frac{1}{\pi} \int_{y_2}^{\pi} \exp(\frac{1}{x_0}(\frac{\pi^2}{6} - 1 + \frac{\pi}{2c_0}) + nx_0)dy \le$$

$$\le \exp(\frac{\pi}{\sqrt{3}} \sqrt{n} - \frac{2\sqrt{3}}{\pi} \sqrt{n}(1 - \frac{\pi^2}{12} - \frac{\pi}{2c_0})) =$$

$$= o(1)2^{-k} q(n)$$

owing to (5.8),(5.9) and (5.12). This completes the proof of our Lemma.

Let

$$(5.13) \qquad (2^8 \le) k_0 \le \mu \le n^{1/6}, \quad \varepsilon = \varepsilon(k_0), \quad 0<\varepsilon<1.$$

We are going to prove the *uniform* estimation

$$|\alpha'_\mu - 2\mu| \le \varepsilon\mu$$

for a number of unequal partitions of n. Here, the number of the *exceptional* partitions is

$$\leq \sum_{k_0 \leq \mu \leq n^{1/6}} E(\mu, n),$$

where $E(\mu, n)$ denotes the number of unequal partitions of n with

(5.14) $$|\alpha'_\mu - 2\mu| > \varepsilon\mu.$$

By (4.22)-(4.23),

(5.15) $$S(n, \Pi^*, [3n^{1/6}]) =$$

$$= (\tfrac{1}{2} + o(1))3n^{1/6} > n^{1/6}$$

holds for all but $O(q(n)n^{-1})$ unequal partitions of n. From (5.15),

(5.16) $$\alpha'_{[n^{1/6}]} \leq 3n^{1/6} .$$

(5.13) and (5.16) imply that

(5.17) $$E(\mu, n) \leq$$

$$\leq \sum_{\alpha'_\mu=\mu}^{[(2-\varepsilon)\mu]} \sum_{1\leq\alpha'_1<\dots<\alpha'_{\mu-1}\leq\alpha'_{\mu-1}}$$

$$q_{\alpha'_\mu}(n-\alpha'_1-\dots-\alpha'_{\mu-1}-\alpha'_\mu) +$$

$$+ \sum_{\alpha'_\mu=[(2+\varepsilon)\mu]+1}^{[3n^{1/6}]} \sum_{1\leq\alpha'_1<\dots<\alpha'_{\mu-1}\leq\alpha'_\mu-1}$$

$$q_{\alpha'_\mu}(n-\alpha'_1-\dots-\alpha'_{\mu-1}-\alpha'_\mu) +$$

$$+ O(q(n)n^{-1}).$$

Here

$$(n-\alpha'_1-\dots-\alpha'_{\mu-1}-\alpha'_\mu)^{1/5} \geq (n-\mu\alpha'_\mu)^{1/5} \geq$$

$$\geq n^{1/5}(1-3n^{-2/3})^{1/5} \geq 3n^{1/6} \geq \alpha'_\mu .$$

We can apply our Lemma and obtain that

$$q_{\alpha'_\mu}(n-\alpha'_1-\dots-\alpha'_{\mu-1}-\alpha'_\mu) =$$

$$O(2^{-\alpha'_\mu})q(n-\alpha'_1-\ldots-\alpha'_{\mu-1}-\alpha'_\mu) =$$

$$= O(2^{-\alpha'_\mu})(n-\alpha'_1-\ldots-\alpha'_{\mu-1}-\alpha'_\mu)^{-3/4} \times$$

$$\times \exp(\frac{\pi}{\sqrt{3}}(n-\alpha'_1-\ldots-\alpha'_{\mu-1}-\alpha'_\mu)^{1/2}) =$$

$$= O(2^{-\alpha'_\mu})n^{-3/4} \exp(\frac{\pi}{\sqrt{3}} \sqrt{n}) =$$

$$= O(2^{-\alpha'_\mu})q(n).$$

From (5.17),

$$E(\mu,n) =$$

$$= O(q(n)) \sum_{\alpha'_\mu=\mu}^{[(2-\varepsilon)\mu]} \binom{\alpha'_\mu-1}{\mu-1} 2^{-\alpha'_\mu} +$$

$$+ O(q(n)) \sum_{\alpha'_\mu=[(2+\varepsilon)\mu]+1}^{[3n^{1/6}]} \binom{\alpha'_\mu-1}{\mu-1} 2^{-\alpha'_\mu} +$$

$$+ O(q(n)n^{-1}),$$

i.e.,

(5.18) $E(\mu, n) =$

$$= O(g(n))\{n^{-1} + \sum_{j=\mu}^{[(2-\varepsilon)\mu]} \binom{j-1}{\mu-1} 2^{-j} +$$

$$+ \sum_{j=[(2+\varepsilon)\mu]+1}^{[3n^{1/6}]} \binom{j-1}{\mu-1} 2^{-j}\}.$$

Clearly,

$$\sum_{j=\mu}^{[(2-\varepsilon)\mu]} \binom{j-1}{\mu-1} 2^{-j} \le \sum_{j=\mu}^{[(2-\varepsilon)\mu]} \binom{j}{\mu} 2^{-j} =$$

$$= 2^{-\mu} + \sum_{\varepsilon\mu \le t \le \mu-1} \binom{2\mu-t}{\mu} 2^{-(2\mu-t)}.$$

Using Stirling's formula we get

$$\log\{\binom{2\mu-t}{\mu} 2^{-(2\mu-t)}\} =$$

$$= \log(2\mu-t)! - \log \mu! - \log(\mu-t)! - (2\mu-t)\log 2 =$$

$$= (2\mu-t)\{\log(2\mu-t)-\log 2\} - \mu \log \mu -$$

$$- (\mu-t)\log(\mu-t) + \frac{1}{2} \log \frac{2\mu-t}{\mu(\mu-t)} + O(1) =$$

$$= \mu \log \frac{\mu-t/2}{\mu} + (\mu-t)\log \frac{\mu-t/2}{\mu-t} +$$

$$+ \frac{1}{2} \log\{\frac{1}{\mu}(1 + \frac{\mu}{\mu-t})\} + o(1) =$$

$$= -\mu \log \frac{1}{1 - \frac{t}{2\mu}} + (\mu-t)\log(1 + \frac{t}{2(\mu-t)}) +$$

$$+ \frac{1}{2} \log(\frac{1}{\mu-t} + \frac{1}{\mu}) + o(1) \le$$

$$\le -\mu(\frac{t}{2\mu} + \frac{1}{2}(\frac{t}{2\mu})^2) + (\mu-t) \frac{t}{2(\mu-t)} + \frac{1}{2\mu} + o(1) =$$

$$= - \frac{t^2}{8\mu} + o(1).$$

Thus,

$$\sum_{j=\mu}^{[(2-\varepsilon)\mu]} \binom{j-1}{\mu-1} 2^{-j} = 2^{-\mu} +$$

$$+ o(1) \sum_{\varepsilon\mu \le t \le \mu-1} \exp(-t^2/8\mu) =$$

$$= o(1) \sum_{t \ge \varepsilon\mu} \exp(-t^2/8\mu) =$$

$$= o(1) \sum_{t \ge \varepsilon\mu} \exp(-\varepsilon t/8) =$$

$$= O(\varepsilon^{-1})\exp(-\varepsilon^2\mu/8).$$

Similarly,

$$\sum_{j=[(2+\varepsilon)\mu]+1}^{[3n^{1/6}]} \binom{j-1}{\mu-1} 2^{-j} < \sum_{j=[(2+\varepsilon)\mu]+1}^{\infty} \binom{j}{\mu} 2^{-j} =$$

$$= \sum_{t>\varepsilon\mu} \binom{2\mu+t}{\mu} 2^{-(2\mu+t)} =$$

$$= \sum_{t>\varepsilon\mu} \exp\{\log(2\mu+t)!-\log \mu!-$$

$$- \log(\mu+t)!-(2\mu+t)\log 2\} =$$

$$= \sum_{t>\varepsilon\mu} \exp\{(2\mu+t)\log(\mu+t/2)-\mu \log \mu- (\mu+t) -\log(\mu+t)-$$

$$- \frac{1}{2} \log \mu + \frac{1}{2} \log \frac{2\mu+t}{\mu+t} + O(1)\} =$$

$$= \sum_{t>\varepsilon\mu} \exp\{\mu \log(1 + \frac{t}{2\mu})-(\mu+t)\log \frac{1}{1 - \frac{t}{2(\mu+t)}} -$$

$$- \frac{1}{2} \log \mu + \frac{1}{2} \log(1 + \frac{\mu}{\mu+t}) + O(1)\} =$$

$$= O(\mu^{-1/2}) \sum_{t>\varepsilon\mu} \exp\{\mu \frac{t}{2\mu} - (\mu+t) \times$$

$$\times \left(\frac{1}{2(\mu+t)} + \frac{1}{2}\left(\frac{t}{2(\mu+t)}\right)^2\right)\} =$$

$$= O(\mu^{-1/2}) \sum_{t>\varepsilon\mu} \exp\left(-\frac{t^2}{8(\mu+t)}\right) =$$

$$= O(\mu^{-1/2}) \sum_{t>\varepsilon\mu} \exp(-\varepsilon t/16) =$$

$$= O(\varepsilon^{-1}\mu^{-1/2})\exp(-\varepsilon^2\mu/16).$$

From (5.18),

$$E(\mu, n) =$$

$$= O(q(n))\{n^{-1}+\varepsilon^{-1}\exp(-\varepsilon^2\mu/8) +$$

$$+ \varepsilon^{-1}\mu^{-1/2}\exp(-\varepsilon^2\mu/16)\}.$$

Finally,

$$\sum_{k_0\le\mu\le n^{1/6}} E(\mu, n) =$$

$$= O(q(n))\{n^{-5/6}+\varepsilon^{-1}\sum_{\mu=k_0}^{\infty} \exp(-\varepsilon^2\mu/8) +$$

$$+ \varepsilon^{-1} \sum_{\mu=k_0}^{\infty} \mu^{-1/2} \exp(-\varepsilon^2 \mu/16)\} =$$

$$= O(q(n))\{n^{-5/6} + O(\varepsilon^{-3})\exp(-\varepsilon^2 k_0/8) +$$

$$+ O(\varepsilon^{-3} k_0^{-1/2})\exp(-\varepsilon^2 k_0/16)\}.$$

Let

$$\varepsilon = \sqrt{\frac{40 \log k_0}{k_0}} \quad . \tag{5.19}$$

For $k_0 \geq 2^8$, we have $0<\varepsilon<1$ and

$$\sum_{k_0 \leq \mu \leq n^{1/6}} E(\mu, n) = O(q(n) k_0^{-3/2}).$$

Consequently, for $2^8 \leq k_0 \leq \mu \ldots^{1/6}$, the *uniform* estimation

$$|\alpha'_\mu - 2\mu| \leq \mu \sqrt{\frac{40 \log k_0}{k_0}}$$

holds for all but $O(q(n) k_0^{-3/2})$ unequal partitions of n.

REFERENCES

[1] Erdős, P. and Lehner, J., The distribution of the

number of summands in the partitions of a positive integer, *Duke Math. Journal*, 8 (1941), 335-345.

[2] Erdős,P. and Szalay, M., On some problems of J.Dénes and P. Turán; in: *Studies in pure mathematics to the memory of Paul Turán*, to appear.

[3] Hardy, G.H. and Ramanujan, S., Asymptotic formulae in combinatory analysis, *Proc. of the London Math. Soc.* (2), XVII (1918), 75-115.

[4] Szalay, M., A note on the dimensions of representations of S_n, *Coll. Math.Soc.János Bolyai* 13. *Topics in Number Theory* (Debrecen, 1974), Ed. P.Turán, 383-388, North-Holland/Elsevier, Amsterdam-New York, 1976.

[5] Szalay, M. and Turán, P., On some problems of the statistical theory of partitions with application to characters of the symmetric group, I, *Acta Math.Acad. Sci.Hungar.*, 29 (1977), 361-379.

[6] Szalay, M. and Turán, P., On some problems of the statistical theory of partitions with application to characters of the symmetric group, II, *Acta Math. Acad.Sci.Hungar.*, 29 (1977), 381-392.

[7] Szalay, M. and Turán, P., On some problems of the statistical theory of partitions with application to characters of the symmetric group, III, *Acta Math. Acad.Sci.Hungar.*, 32 (1978), 129-155.

P. ERDŐS

MATHEMATICAL INSTITUTE OF THE

HUNGARIAN ACADEMY OF SCIENCES

H-1053 BUDAPEST, REÁLTANODA U.13-15.

HUNGARY

M. SZALAY

DEPARTMENT OF ALGEBRA AND NUMBER THEORY

EÖTVÖS LORÁND UNIVERSITY

H-1088 BUDAPEST, MUZEUM KÖRUT 6-8.

HUNGARY