ON SOME PROBLEMS IN GRAPH THEORY, COMBINATORIAL ANALYSIS AND COMBINATORIAL NUMBER THEORY

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ABSTRACT In this paper I discuss some old problems, most of them due to myself and my collaborators. Many of these problems have been undeservedly neglected, even by myself. I also discuss some new problems.

I. Graph theory

1. $G(n)$ is a graph of n vertices and $G(n; e)$ is a graph of n vertices and e edges. Is it true that if every induced subgraph of a $G(10n)$ of $5n$ vertices has more than $2n^2$ edges then our $G(10n)$ contains a triangle? It is easy to show that if true this result is best possible. To see this let A_i, $i=1$, $2, \ldots, 5$, be sets of $2n$ vertices, put $A_1 = A_6$ and join every vertex of A_i to every vertex of A_{i+1}. This $G(10n; 20n^2)$ has of course no triangle and every induced subgraph of $5n$ vertices contains at least $2n^2$ edges. Equality is of course possible: choose A_1, A_3 and half the vertices of A_4.

Simonovits pointed out to me that a graph of completely different structure also shows that the conjecture, if true, is best possible. Consider the Petersen graph, which is a $G(10; 15)$. Replace each vertex by a set of n vertices and replace every edge of the Petersen graph by the n^2 edges of a $K(n, n)$. This gives a $G(10n; 15n^2)$ and it is easy to see that every induced subgraph of $5n$ vertices has at least $2n^2$ edges.

The fact that two graphs of different structure are extremal perhaps indicates that the conjecture is either false or difficult to prove. I certainly hope that the latter is the case.

It is perhaps tempting to conjecture that my graph has the following extremal property. If a $G(10n)$ has no triangle and every induced subgraph of $5n$ vertices has at least $2n^2$ edges, then our graph can have at most $20n^2$ edges. Perhaps the graph of Simonovits has the smallest number of edges among all extremal graphs; perhaps in fact these two graphs are the only extremal graphs.

Many generalizations are possible; the triangle could be replaced by other graphs. Is it true that every $G((4h+2)n)$, every induced subgraph

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of $(2h+1)n$ vertices of which has more than $2n^2$ edges, contains an odd circuit of fewer than $2h+1$ edges?

Let $0<\alpha<1$. Determine the smallest c_α so that if every induced subgraph of $G(n)$ of $\lfloor \alpha n \rfloor$ vertices contains more than $c_\alpha n^2$ edges then $G(n)$ has a triangle. My conjecture was that $c_{1/2}=\frac{1}{50}$.

Observe that it is not difficult to prove that for $n>n_0(r)$ every $G(n;cn^2)$ contains a $K(r)$, provided that every induced subgraph of $n/2$ vertices has $(1+o(1))(cn^2/4)$ edges.

2. Is it true that every graph of $5n$ vertices which contains no triangle can be made bipartite by the omission of at most n^2 edges? The same graph as used in paragraph 1 shows that the conjecture, if true, is best possible. (Replace each vertex of a pentagon by n vertices, etc.)

Is it true that a $G(5n)$ which has no triangle contains at most n^5 pentagons? Again the same graph shows that, if true, this is the best possible. Here also many generalizations are possible.

3. Let G be a bipartite graph of n white and $n^{2/3}$ black vertices. Is it true that if our graph has more than cn edges (where c is a sufficiently large constant), then it contains a C_6? It is easy to see that it must contain a C_8 but does not have to contain a C_4. Simonovits strongly disbelieves this conjecture and I have no real evidence for its truth. It is easy to see that this conjecture, if true, is best possible. To see this observe that Benson's graph has $cn^{4/3}$ edges and n black and n white vertices, thus a suitable subgraph has n white and $n^{2/3}$ black vertices and cn edges.

C. Benson, Minimal regular graphs of girth eight and twelve. *Canad. J. Math.* **18** (1966), 1091–1094.

4. Several hundred papers have been published on extremal graph theory and recently Bollobás published an excellent book on the subject. Here I state only a few problems which have not been thoroughly investigated. First a recent problem of Simonovits and myself.

Denote by $f(n;H)$ the smallest integer such that every graph $G(n;f(n;H))$ contains H as a subgraph. Turán, who started extremal graph theory, determined $f(n;K(r))$ for every r. In particular he proved that $f(n,K(3))=\lfloor n^2/4 \rfloor+1$.

Rademacher was the first to observe that every $G(n;\lfloor n^2/4 \rfloor+1)$ contains at least $\lfloor n/2 \rfloor$ triangles. This result was generalized and extended by Bollobás, Lovász, Simonovits and myself. Simonovits and I asked: Is it true that every $G(n;f(n;C_4))$ contains at least $(1+o(1))\cdot n^{1/2}$ C_4s. We could not even prove that the number of these C_4s tends to infinity.

Brown, V. T. Sós, Rényi and I proved that $f(n; C_4) = (\frac{1}{2} + o(1))n^{3/2}$ and (p is a power of a prime) that

$$f(p^2 + p + 1, C_4) \geqslant \tfrac{1}{2}p(p+1)^2. \tag{1}$$

Is it true that there is equality in (1)? Füredi proved this for $p = 2^k$, and very recently he proved that there is always equality in (1). I observed that for every n,

$$f(n; C_4) \leqslant \tfrac{1}{2}n^{3/2} + (1 + o(1))\frac{n}{4}, \tag{2}$$

and conjectured that there is equality in (2).

Kövári, V. T. Sós and Turán proved that

$$f(n; K(r, r)) < c_r n^{2-(1/r)}. \tag{3}$$

Very probably (3) is best possible for every r. This problem is still open for $r > 3$. Very recently Frankl proved that

$$f(n; K(r; 2^r)) > c'_r n^{2-(1/r)}.$$

I conjectured some time ago that for every $\varepsilon > 0$ there is a c_ε such that every $G(n; \lfloor n^{1+\varepsilon} \rfloor)$ contains a non-planar subgraph of at most c_ε vertices.

Dirac asked in a conversation: Let m and n be fixed. How many edges can have a $G(n)$ if it does not contain a saturated planar subgraph of at least m vertices? Simonovits nearly completely solved this problem.

Denote by $f(n, m)$ the smallest integer such that every $G(n; f(n; m))$ contains all saturated planar graphs of m vertices. The four-colour theorem easily implies $f(n; m) = (1 + o(1))(n^2/3)$, but it might be of interest to get a sharper formula which would also show the dependence on m.

Simonovits and I posed the following problem: Is it true that

$$\lim f(n; H)/n^{3/2} = \infty \tag{4}$$

holds if and only if H contains a subgraph H', each vertex of which has degree greater than 2?

This attractive conjecture is very far from being settled. The following two graphs perhaps could give a counterexample. Define H_k as follows: The vertices of H_k are $x; y_1, \ldots, y_k, z_1, z_2, \ldots, z_{\binom{k}{2}}$. The vertex x is joined to all the y_js and each z_i is joined to two y_js (no two z_is are joined to the same pair). Is it true that

$$f(n; H_k) < cn^{3/2}? \tag{5}$$

I proved (5) for $k = 3$ but for $k > 3$ I do not know if (5) holds. The second graph H was once considered by V. T. Sós, Simonovits and myself. H has five black and five white vertices, x_1, x_2, a, x_3, x_4 and y_1, y_2, b, y_3, y_4

respectively; a is joined to every y_j and b to every x_i; a and b are not joined. $(x_1, x_2; y_1, y_2)$ and $(x_3, x_4; y_3, y_4)$ form a C_4. Is it true that $f(n; H) < cn^{3/2}$?

In a recent interesting paper Faudree and Simonovits obtain several striking new results in extremal graph theory. They hope that a further development of their method will lead to the construction of two graphs H_1 and H_2 for which

$$\lim_{m \to \infty} f(n; H_1; H_2)/\min\{f(n; H_1), f(n; H_2)\} = 0.$$

Here $f(n; H_1, H_2)$ is the smallest integer for which every $G(n; f(n; H_1, H_2))$ contains either H_1 or H_2.

B. Bollobás, *Extremal Graph Theory*, London Mathematical Society Monographs no. 11, Academic Press, London, 1978.

P. Erdös, On a theorem of Rademacher and Turán. *Illinois J. Math.* **6** (1962), 122–127.

B. Bollobás, Relations between sets of complete subgraphs. In *Fifth British Combinatorial Conference*, Utilitas Mathematicae, Winnipeg, 1976, pp. 161–170.

L. Lovász and M. Simonovits, On the number of complete subgraphs of a graph. In *Fifth British Combinatorial Conference*, Utilitas Mathematicae, Winnipeg, 1976, pp. 431–441.

P. Erdös, A. Rényi and V. T. Sós, On a problem of graph theory. *Studia Sci. Math. Hungar.* **1** (1966), 215–235.

W. Brown, On graphs that do not contain a Thomsen graph. *Canad. Math. Bull.* **9** (1966), 281–285.

Z. Füredi, Graphs without quadrilaterals. *J. Combinat. Theory* **34** (1983), 187–190.

T. Kövári, V. T. Sós and P. Turán, On a problem of K. Zarankiewicz. *Colloq. Math.* **3** (1959), 50–57.

M. Simonovits, On graphs not containing large saturated planar graphs. In *Infinite and Finite Sets*, Coll. Math. Soc. J. Bolyai no. 10, North-Holland, Amsterdam, 1975, pp. 1365–1386.

P. Erdös, On some extremal problems in graph theory. *Israel J. Math.* **3** (1965), 113–116.

R. Faudree and M. Simonovits, On a class of degenerate extremal graph problems. *Combinatorica* **3** (1983), 83–94.

P. Erdös and M. Simonovits, Supersaturated graphs. *Combinatorica* **3** (1983), in press.

5. Many papers and the excellent book of Graham, Rothschild and Spencer have recently been published on Ramsey theory. Here I only want to say a few words and mention a new problem of Hajnal and myself. Let G_1 and G_2 be two graphs, and denote by $r(G_1, G_2)$ the smallest integer m for which, if we colour the edges of $K(m)$ by two

colours, I and II, then either there is a G_1 all of whose edges have colour I or a G_2 all of whose edges have colour II. Graver and Yackel, Ajtai, Komlós and Szemerédi and I myself proved that

$$c_2 n^2/(\log n)^2 < r(K(n), K(3)) < c_1 n^2/\log n.$$

I always conjectured that

$$r(K(n), C_4) < n^{2-\varepsilon}. \tag{6}$$

Many colleagues doubt whether (6) holds. As far as I know everybody believes that

$$r(K(n), C_4)/r(K(n), K(3)) \to 0. \tag{7}$$

Szemerédi recently observed that

$$r(K(n), C_4) < cn/(\log n)^2 < cr(K(n), K(3)).$$

Denote by $R(G_1, G_2)$ the smallest integer m for which there is a $G(m)$ with the property that if we colour the edges of G with two colours I and II in an arbitrary way, then either there is an *induced* subgraph G_1 of G, all of whose edges have colour I, or an induced subgraph G_2, all of whose edges have colour II. The existence of $R(G_1, G_2)$ is not at all obvious. As far as I know this was first conjectured by Hansen and proved simultaneously by Deuber, Rödl, and Erdös, Hajnal and Pósa. Hajnal and I observed that if G_1 and G_2 have at most n vertices then

$$R(G_1, G_2) = m < 2^{2^{n^{1+\varepsilon}}}. \tag{8}$$

We have never published the not entirely trivial proof of (8) since Hajnal and I thought that perhaps

$$\max R(G_1, G_2) = r(K(n), K(n)). \tag{9}$$

Conjecture (9) is perhaps a little too optimistic, but we have no counterexample. Perhaps there is a better chance to prove $R(G_1, G_2) < 2^{cn}$.

R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, John Wiley, New York, 1980.

W. Deuber, A generalization of Ramsey's theorem. In *Infinite and Finite Sets*, Coll. Math. Soc. J. Bolyai no. 10, North-Holland, Amsterdam, 1975, pp. 323–332.

P. Erdös, A. Hajnal and L. Pósa, Strong embeddings of graphs into coloured graphs. In *Infinite and Finite Sets*, Coll. Math. Soc. J. Bolyai no. 10, North-Holland, Amsterdam, 1975, pp. 585–596.

V. Rödl, A generalization of Ramsey's theorem. In *Hypergraphs and Block Systems*, Zielona Gora, 1976, pp. 211–220.

P. Erdös, Graph theory and probability II. *Canad. J. Math.* **13** (1961), 364–352.

J. E. Graver and I. Yackel, Some graph theoretic results associated with Ramsey's theorem. *J. Combinat. Theory* **4** (1968), 125–175.
M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers. *J. Combinat. Theory* A **29** (1980), 354–360.

6. During a recent visit to Prague, I learned of the following surprising result of Frankl and Rödl which settled an old conjecture of mine. Let $G_i^r(n_i; e_i)$ be a sequence of r-graphs. I say that their density is α if α is the largest number for which there is a sequence of induced subgraphs $G_i^{(r)}(m_i; e_i')$ of $G_i^{(r)}(n_i; e_i)$ with $m_i \to \infty$ and

$$e_i' = (\alpha + o(1))\binom{m_i}{r} \quad \text{as} \quad i \to \infty.$$

In a slightly imprecise but more illuminating way we can say that a large r-graph $G^{(r)}(n; e)$ has density α is α is the largest number for which there is a large subgraph $G(m_i; e_i')$ (of m_i vertices and e_i' edges) for which

$$e_i' = (\alpha + o(1))\binom{m_i}{r}.$$

A well known theorem of Stone and myself asserts that for $r = 2$ the only possible values of the densities of an ordinary graph are $1 - 1/r$, $1 \leq r \leq \infty$. (See also some papers of Bollobás, Chvátal, Simonovits, Szemerédi and myself.)

I conjectured that for $r > 2$ the set of possible densities forms a well ordered set. This was disproved by Frankl and Rödl. At this moment it is not yet clear what are the possible values of the densities of r-graphs for $r > 2$. The results of Frankl and Rödl will be soon published in *Combinatorica.*

P. Erdös and A. H. Stone, On the structure of linear graphs. *Bull. Amer. Math. Soc.* **52** (1946), 1087–1091.
P. Erdös and M. Simonovits, A limit theorem in graph theory. *Studia Sci. Math. Hungar. Acad.* **1** (1966), 51–57.
B. Bollobás and P. Erdös, On the structure of edge graphs. *Bull. London Math. Soc.* **5** (1973), 317–321.
B. Bollobás, P. Erdös and M. Simonovits, On the structure of edge graphs II. *J. London Math. Soc.* **12** (1976), 219–224.
V. Chvátal and E. Szemerédi, On the Erdös–Stone theorem. *J. London Math. Soc.* **23** (1981), 193–384.
P. Erdös, On extremal problems of graphs and generalized graphs. *Israel J. Math.* **2** (1965), 183–190.

7. Frankl and Rödl also settled a very recent question of Nešetřil and myself. At the recent meeting at Poznan on random graphs Nešetřil and I

conjectured that for every $\varepsilon>0$ there is a graph $G(n;e)$ which contains no $K(4)$ but every subgraph of $(\frac{1}{2}+\varepsilon)e$ edges of it contains a triangle. This conjecture was settled by Frankl and Rödl by the probability method early in September 1983. Many generalizations are possible which are not yet completely cleared up.

The graph of Frankl and Rödl has n vertices and $n^{3/2-\varepsilon}$ edges. Frankl and Rödl also proved that there is no $c>0$ such that for every $\varepsilon>0$ there is such a graph with at least cn^2 edges. Their method at present does not seem to work if $K(3)$ is replaced by $K(r)$, $K(4)$ by $K(r+1)$ and $(\frac{1}{2}+\varepsilon)$ by $1-(1/r)+\varepsilon$.

8. With Nešetřil we posed the following problem: Is there a G whose every edge is contained in at most three triangles and for which $G\to(K(3),K(3))$ holds? (In other words, if we colour the edges of G by two colours then is there always a monochromatic triangle?) Observe that since $K(6)\to(K(3),K(3))$ "three" cannot be replaced by "four". Let $G(n)\to(K(3),K(3))$ critically (i.e. if we omit any edge of $G(n)$ then this property no longer holds). Nešetřil and Rödl proved that such graphs $G(n)$ exist. We thought that for every k there is an n_k so that if $n>n_k$ and $G(n)\to(K(3),K(3))$ critically then there is an edge of $G(n)$ which is contained in at least k triangles. Nešetřil has just proved that this is not so.

J. Nešetřil and V. Rödl, The structure of critical Ramsey graphs. *Acta Math. Acad. Sci. Hungar.* **32** (1978), 295–300.

9. Let $G(n)$ be a connected graph of n vertices. Denoted by $f(G)$ the smallest integer for which the vertices of G can be covered by $f(G)$ cliques and let $h(G)$ be the largest integer for which there are $h(G)$ edges no two of which belong to the same clique. Parthasaraty and Choudum conjectured that $h(G)\geq f(G)$. By probabilistic methods I disproved this conjecture. In fact I showed that there is a $G(n)$ for which

$$f(G(n))>\frac{c_1 n}{(\log n)^3}h(G(n)). \tag{10}$$

I conjectured that (10) is best possible, i.e. that for every $G(n)$

$$f(G(n))<\frac{c_2 n}{(\log n)^3}h(G(n)). \tag{11}$$

I had difficulties in proving (11), which as far as I know is still open. I think the proof of (11) will not be difficult and that I am perhaps overlooking a simple argument. I thought that if $h(G(n))$ is small then

$h(G(n)) = f(G(n))$ is true. This was known for $n = 3$ and Kostochka just informs me that if $h(G) \leq 5$ then $f(G) = h(G)$, but there is a $G(n)$ for which $h(G(n)) = 6$ and $f(G(n)) > n^{1/17}$. Kostochka's nice result answers my question very satisfactorily; nevertheless many questions remain, e.g. is it true that if $f(G(n)) > n^{1-\varepsilon}$, $\varepsilon \to 0$, then $h(G(n)) \to \infty$?

An old conjecture of Hajós stated: Is it true that every k-chromatic graph contains a topological complete h-gon; i.e. it contains h vertices every two of which can be joined by paths every two of which are disjoint (except having common end-points)? This was disproved by Catlin and Kostochka. Fajtlowicz and I showed by probabilistic methods that if $K(G(n))$ is the chromatic number of G and $T(G)$ is the number of vertices of the largest topologically complete graph embedded in G, then for almost all graphs

$$K(G(n)) > c_1\left(\frac{n^{1/2}}{\log n}\right)T(G). \tag{12}$$

Perhaps (12) is best possible, i.e.

$$K(G(n)) < c_2\left(\frac{n^{1/2}}{\log n}\right)T(G).$$

Bollobás, Catlin and I, on the other hand, proved that Hadwiger's conjecture holds for almost all graphs $G(n)$.

P. Erdös, [Title unknown.] *J. Math. Res. Exposition* **2** (1982), 93–96.
P. Erdös and S. Fajtlowicz, On the conjecture of Hajós. *Combinatorica* **1** (1981), 141–143.
P. A. Catlin, Hajós's graph-colouring conjecture: variations and counterexamples. *J. Combinat. Theory B* **26** (1979), 268–274.
B. Bollobás, P. A. Catlin and P. Erdös, Hadwiger's conjecture is true for almost every graph. *Europ. J. Combinat.* **1** (1980), 195–199.

10. Denote by $C^{(n)}$ the graph of the n-dimensional cube. $C^{(n)}$ has 2^n vertices and $n2^{n-1}$ edges. How many edges of $C^{(n)}$ ensure the existence of a C_{2r}, a circuit of size $2r$? Perhaps for $r > 2$, $o(n2^n)$ edges suffice. Perhaps $(\frac{1}{2}+\varepsilon)n2^{n-1}$ edges suffice for a C_4. It is easy to see that $\frac{1}{2}n2^{n-1}$ do not suffice for a C_4 and Chung observed that $\frac{2}{3}n2^{n-1}$ suffice for a C_4. The following is an old and forgotten problem of Graham and myself. It is well known that $K(2^n)$ can be decomposed as the union of n bipartite graphs, but $K(2^n+1)$ can not be so decomposed. Suppose we decompose $K(2^n+1)$ as the union of n graphs. What is the least odd circuit which must occur in any of the decompositions? To be precise, denote by $f(n)$ the smallest integer such that if

$$K(2^n) = \bigcup_{i=1}^{n} G_i$$

then one of the G_is contains an odd circuit of order at most $f(n)$. How large is $f(n)$? As far as I know, nothing is known about this problem, which has been completely forgotten or at least neglected, perhaps undeservedly so.

P. Erdös and R. L. Graham, On partition theorems for finite graphs. In *Infinite and Finite Sets*, Coll. Math. Soc. J. Bolyai no. 10, North Holland, Amsterdam, 1975, pp. 515–528.

11. The following problem is due to Fajtlowicz and myself. A graph $G(n)$ is said to have property $I_{r,l}$ (respectively $I_{r,\infty}$) if every set of l independent vertices (respectively every set of independent vertices) has a common neighbour and $G(n)$ contains no $K(r)$. It is said to have property I_∞ if we only assume that every set of independent vertices has a common neighbour. $f(n;r,l)$ (respectively $f(n;r,\infty)$ or $f(n;\infty)$) is the largest integer such that every graph with property $I_{r,l}$ (respectively $I_{r,\infty}$ or I_∞) has a vertex of degree greater than or equal to $f(n;r,l)$ (respectively $f(n;r,\infty)$ or $f(n;\infty)$). Determine or estimate $f(n;r,l)$, $f(n;r,\infty)$ and $f(n;\infty)$ as well as possible. Pach proved that $f(n;3,\infty)=(n+1)/3$ and $f(n;3,3)<n^{1-c_{3,3}}$, where $c_{3,3}$ is a positive constant. Pach and I have just proved that $f(n;r,r)<n^{1-c_{r,r}}$ and by the probability method we established

$$f(n;\infty)=(1+o(1))\frac{n\log\log n}{\log n}.$$

We do not know if $f(n;3,l)>cn$ holds for some $l>3$ or $f(n;r,\infty)>cn$ is true for some $r>3$.

The following slightly modified problem could also be considered. Assume that a graph $G(n)$ is such that every set of r vertices of $G(n)$ has a common neighbour. Then of course it is immediate that our $G(n)$ must contain a $K(r+1)$. On the other hand it is easy to see by the probability method that there is such a graph which contains no $K(r+2)$ and each vertex of which has degree $o(n)$.

J. Pach, Graphs whose every independent set has a common neighbour. *Discrete Math.* **37** (1981), 217–218.

12. A group is said to have property A_k if it has at most k elements which pairwise do not commute. About 10 years ago I asked: Determine or estimate the smallest $f(k)$ so that every group with property $A(k)$ is the union of $f(k)$ or fewer Abelian groups. This is a finite modification of a problem considered by Bernhard Neumann. Isaacs proved that

$$(1+c)^k<f(k)<k!^{2+\varepsilon}.$$

13. Finally I state a few recent problems which have not been investigated carefully.

Let G be a graph. Denote by $f_b(G)$ the maximal number of edges of a bipartite subgraph contained in G and by $f_3(G)$ the maximal number of edges of a triangle-free graph contained in G. Trivially,

$$f_3(G) \geq f_b(G). \tag{13}$$

Can one characterize the graphs for which there is equality in (13)? Or at least can one give a fairly general class of graphs having this property? Clearly $K(n)$ has equality in (13) by Turán's theorem and Simonovits easily showed by the method of Zykov that all complete r-partite graphs also have equality in (13). V. T. Sós asked: Let G be such that every circuit has a diagonal (or perhaps only every odd circuit has a diagonal). Is it then true that G has equality in (13)?

Horak, Kratochvil and I considered the following questions. Let $G_1(n), \ldots, G_r(n)$ be edge-disjoint subgraphs of $K(n)$ such that every Hamiltonian circuit of $G(n)$ has an edge in common with the $G_i(n)$. Put

$$f(n; r) = \min \sum_{i=1}^{r} e(G_i(n)),$$

where $e(G)$ denotes the number of edges of G. It is easy to see that $f(n; r) = r(n-2)$ for $r \leq 3$ and $f(n; r) < 2^{c_1 r}$ for $r > 3$. Can one determine the exact value of $f(n; r)$ for $r > 3$? What is the largest value $g(n)$ of r for which such graphs $G_i(n)$ exist? Similar questions can be asked for Hamiltonian paths or other subgraphs of $K(n)$.

Shortly after we posed these questions. Horák and Širán succeeded in determining both $f(n; r)$ and $g(n)$. They proved that

$$g(n) = \left\lfloor 3 + \log_2 \frac{n-1}{3} \right\rfloor \quad \text{for} \quad n \geq 4.$$

Furthermore, setting $w(n, r) = 3 \cdot 2^{r-4}(2n - 3 \cdot 2^{r-3} - 1)$,

$$f(n; r) = w(n, r) \qquad \text{for} \quad 4 \leq r \leq 3 + \log_2 \frac{n + b_n}{9},$$

$$f(n; r) = w(n, r-1) + c_n \quad \text{for} \quad 3 + \log_2 \frac{n + b_n}{9} < r \leq g(n),$$

where $c_n = (n^2 - 1)/8$. $b_n = 1$ for odd values of n and $c_n = (n^2 + 2n - 8)/8$. $b_n = 4$ for even values of n.

Let G be a graph of e edges. Is it true that

$$r(G, G) < 2^{c_1 e^{1/2}}? \tag{14}$$

If true, (14) is easily seen to be best possible apart from the value of c_1. Probably $r(G)$ is maximal if G is as complete as possible.

V. T. Sós asked: A graph G is said to be Ramsey-critical if

$$r(G, G) > r(G-e, G-e) \tag{15}$$

for every edge of G. Can one characterize the graphs which satisfy (15)? As far as I know this simple and interesting question has not yet been considered. It seems certain that $K(n)$ satisfies (15), but even this is not known and it is not clear to me that (15) holds for almost all graphs. A related question is: Can one characterize the graphs for which

$$r(G, G) < r(G+e, G+e), \tag{16}$$

where e is any edge not in G which joins two vertices of G?

II. Set systems

I have published many problems on set systems, therefore I mention only some problems which are perhaps not too well known.

1. Rado and I once considered the following problem. Determine or estimate the largest integer $g(n)$ for which one can give $g(n)$ sets A_h, $|A_h| = n$, $1 \leqslant h \leqslant g(n)$, so that for every three of our A_hs the union of some two of them contains the third. Jean Larson showed $g(2n) \geqslant (n+1)^2$ and Frankl and Pach proved that $g(2n) = (n+1)^2$. More generally, let $g_t(n)$ denote the maximum number m such that there exists a family of n-sets $\{A_1, A_2, \ldots, A_m\}$ without t disjointly representable members (i.e. any t members of the family contains one which is covered by the remaining $t-1$ members). Frankl and Pach conjecture that

$$g_t(n) = T(n+t-1, t, t-1),$$

where $T(n, k, l)$ denotes the Turán number, the maximum cardinality of a family of l-sets on an n-set without a complete subgraph with k points. They can prove that

$$T(n+t-1, t, t-1) < g_t(n) < \binom{n+t-1}{t-1}.$$

Let $h_t(n)$ denote the maximum cardinality of a (non-uniform) family on n points without t disjointly representable members. A theorem of Sauer implies that

$$h_t(n) \leqslant \sum_{1 \leqslant i \leqslant t-1} \binom{n}{i}.$$

Frankl gave a construction for $t=3$ (unpublished) and Füredi and Quinn proved that equality holds here for all t. There are a large number of optimal families.

Frankl and Pach also conjecture that the following beautiful generalization of the Erdös–Ko–Rado theorem is true. Suppose that $\mathscr{H}$ is a system of k element subsets of an n element set and that

$$|\mathscr{H}| > \binom{n-1}{k-1}.$$

Then there exists a $H_0 \in \mathscr{H}$ such that for every subset $A \subseteq H_0$ one can choose $H \in \mathscr{H}$ satisfying $H \cap H_0 = A$. (We obtain the Erdös–Ko–Rado theorem in the special case $A = \varnothing$.) They can prove this for set-systems of cardinality greater than $\binom{n}{k-1}$ by using a linear algebraic approach.

P. Frankl and J. Pach, On the number of sets in a null-t design. *Europ. J. Combinat.* **4** (1983), 21–23.
P. Frankl and J. Pach, On disjointly representable sets. *Combinatorica* **4** (1984), in press.
Z. Füredi and F. Quinn, Traces of finite sets. *Ars Combinat.* in press.
N. Sauer, On density of families of sets. *J. Combinat. Theory* A **13** (1972), 145–147.
P. Erdös, Chao Ko and R. Rado, Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)* **12** (1961), 313–320.

2. A few years ago Frankl formulated the following interesting problem: Let $\{A_k\}, |A_k| = n$, be a two-chromatic clique, i.e. $|A_{k_1} \cap A_{k_2}| \neq \varnothing$ and $\bigcup_k A_k = S$ can be decomposed into the union of two disjoint sets S_1 and S_2 so that no A_k is contained in S_1 or S_2. Let $f(n)$ be the smallest integer for which there is a set B, $|B| = f(n)$, $A_i \not\subseteq B$, $B \cap A_i \neq \varnothing$. Frankl showed that $n + c\sqrt{n} < f(n) < n^2 \log n$. It would of course be very desirable to have a better estimate for $f(n)$.

3. Let $\{A_i : 1 \leqslant i \leqslant t_n\}$ be a family of n-sets which is a maximal clique, i.e. which is such that $A_i \cap A_j \neq \varnothing$ for all $1 \leqslant i < j \leqslant n$ and there is no n-set B for which $B \cap A_i \neq \varnothing$ for every $1 \leqslant i \leqslant t_n$. Füredi asked whether it is true that

$$\left| \bigcup_{i=1}^{t_n} A_i \right| \leqslant t_n ?$$

If true, this would be a generalization of Fischer's inequality.

Perhaps the conjecture remains true if $|A_i| = n$ is replaced by $|A_i| \leqslant n$.

This, if true, would be a generalization of the theorem of de Bruijn and myself.

N. G. de Bruijn and P. Erdös, On a combinatorial problem. *Indig. Math.* **10** (1948), 421–423.

4. The following problem is due to Duke and myself. Consider set systems $\{A_i: 1 \leq i \leq m\}$, $A_i \subset S$, $|S| = n$, $|A_i| = r \geq 3$, such that no t sets $A_{i_1}, A_{i_2}, \ldots, A_{i_t}$ intersect in the same element (i.e. the family $\{A_i: 1 \leq i \leq m\}$ contains no $\triangle$-systems of size t and kernel 1). Denote by $f(n; r, t)$ the maximal value of m. With Duke we proved that $f(n; r, t) < c(r, t)n^{r-2}$.

Recently Chung and Frankl proved that

$$f(n; 3, t) = 2\binom{t}{2}(n-2t) + 2\binom{t}{3}$$

for $n > n_0(t)$, t odd, and

$$f(n; 3, t) = (t(t-\tfrac{3}{2}) + O(1))(n-2t+1)$$

for t even.

Here is their construction for odd t: Consider two disjoint t element subsets X and Y of S. Our family consists of all three element subsets A_i of S which are disjoint from one of X and Y and intersect the other in at least two elements.

For $r \geq 4$ we can add to the above construction all r sets intersecting both X and Y in at least two elements.

Frankl and Füredi proved that for $n > n_0(r, t)$ this construction is best possible. They also proved that for every c_1 there is a c_2 such that if $r = 5$, $m > c_2 n^2$ then there are c_1 sets which form a $\triangle$-system of kernel 2. It would be desirable to determine c_2 as a function of c_1.

Here I want to remind the reader of one of my favourite problems, asked by Rado and myself more than 20 years ago. Is it true that there is an absolute constant C so that any family of C^n n-sets contains a $\triangle$-system of three sets? I offer US\$1000 for a proof or disproof.

R. Duke and P. Erdös, Systems of finite sets having a common intersection. In *Proceedings of the Southeastern Conference on Combinatorics, etc.*, Congressus Num. XIX, Utilitas Mathematicae, Winnipeg, 1977, pp. 247–252.
P. Erdös and R. Rado, Intersection theorems for systems of sets. *J. London Math. Soc.* **35** (1960), 85–90.

5. Let $|S| = 2n$, $A_k \subset S$, $|A_k| = n$, $1 \leq k \leq T_n$ be a system of subsets of S. Assume that the number of pairs A_i, A_j satisfying $A_i \cap A_j = \emptyset$ is greater

than or equal to 2^{2n}. I conjectured a few years ago that this implies

$$T_n > (1-\varepsilon)2^{n+1}.$$

Frankl recently proved my conjecture. Frankl conjectures that if $T_n > 2^n n^c$ then the number of disjoint pairs is at most $\varepsilon(c)T_n^2$, where $\varepsilon(c) \to 0$ as $c \to \infty$.

Frankl proved that if $2^{1/(s+1)} < a \leq 2^{1/s}$ and $T_n = a^n$ then there are at most

$$\frac{s-1+\varepsilon}{s}\binom{T}{2^n}$$

disjoint pairs. It is easy to see that this result is best possible. In some sense this is a phenomenon similar to the Erdös–Stone theorem.

I hope Frankl will soon publish his interesting results in detail.

6. Let $A_1, \ldots, A_m$, $|A_i| = r$, $1 \leq i \leq m$, $A_i \cap A_j \neq \emptyset$, $1 \leq i < j \leq m$. Assume that for every A_i and every proper subset B of A_i there is an A_j satisfying $A_j \cap B = \emptyset$ (i.e. the family $\{A_i\}$ is a clique but this property gets destroyed if we replace any of the sets by a smaller one). Denote by $f(r)$ the largest possible value of

$$\left|\bigcup_{i=1}^{m} A_i\right|.$$

Calczinska-Karlowitz showed that $f(r)$ is finite. After some results of Ehrenfeucht and Mycielski, we proved with Lovász that

$$\frac{1}{2}\binom{2r-1}{n-1} + 2r - 2 \leq f(r) \leq \frac{r}{2}\binom{2r-1}{r}. \tag{17}$$

By making use of a theorem of Bollobás, Tuza improved (17) to

$$2\binom{2r-4}{r-2} + 2r - 4 \leq f(r) \leq \binom{2r-1}{r-1} + \binom{2r-4}{r-2}. \tag{18}$$

Tuza conjectures that the lower bound in (18) is exact.

M. Calczinska-Karlowitz, Theorem on families of finite sets. *Bull. Acad. Polon. Sci. Ser. Math. Astr. Phys.* **12** (1964), 87–89.

A. Ehrenfeucht and J. Mycielski, Interpolation of functions over a measure space and conjectures about memory. *J. Approximation Theory* **9** (1973), 218–236.

P. Erdös and L. Lovász, Problems and results on three-chromatic hypergraphs and some related questions. In *Infinite and Finite Sets*, Coll. Math. Soc. J. Bolyai no. 10, North Holland, Amsterdam, 1975, pp. 609–627.

B. Bollobás, On generalized graphs. *Acta Math. Acad. Sci. Hungar.* **16** (1965), 447–452.

Z. Tuza, Critical hypergraphs and intersecting set-pair systems. *Discrete Math.* to appear.

7. J. Larson and I asked: Is there an absolute constant C so that for every n there is a partially balanced block design $\{A_i\}$ on a set S of n elements so that

$$|A_i| > n^{1/2} - C \tag{19}$$

holds for every i? Our guess was that such a C does not exist for all n.

I just heard that Shrikande and Singhi have proved that if our guess is wrong, i.e. if a block design exists for every n satisfying (19), then there is a finite geometry whose order is not of the form p^2+p+1, where p is a power of a prime.

As far as I know it is not impossible that there is a sequence $n_1 < n_2 < \ldots$ which is such that for every i we have $n_{i+1} - n_i < c$ and there exists a finite geometry $n_i^2 + n_i + 1$ elements. It is easy to see that in this case there is a block design for every n satisfying (19).

Is there a partially balanced block design on n elements, say $\{A_i: 1 \leq i \leq t_n\}$, so that for every r the number of indices i for which $|A_i| = r$ is less than $cn^{1/2}$? It is easy to see that, if true, then apart from the value of c this result is best possible, though perhaps $cn^{1/2}$ could be replaced by $cr^{1/2}$.

P. Erdös and J. Larson, On pairwise balanced block designs with the sizes of blocks as uniform as possible. *Ann. Discrete Math.* **15** (1983), 129–134.

8. Is it true that in every finite geometry of p^2+p+1 elements there is a blocking set which meets every line in fewer than C points? Bruen and Freeman showed that such geometries exist for infinitely many p. I have been informed that T. Evans stated this problem several years before me.

9. Let $\{A_k\}$ be a family of subsets of a set with n elements. When is there a set S such that

$$1 \leq |A_k \cap S| \leq C \tag{20}$$

for every k? Perhaps there is an S satisfying (20) provided that $|A_k| > c_1 n^{1/2}$ and $|A_{k_1} \cap A_{k_2}| \leq 1$ for all k, k_1 and k_2 with $k_1 \neq k_2$. Maybe the last condition can even be weakened to $|A_{k_1} \cap A_{k_2}| \leq c_2$ with $c = c(c_1, c_2)$.

The following is an old question posed by Grünbaum and myself. Let $\{A_k: 1 \leq k \leq T_n\}$ be a partially balanced block design on n elements. Define a graph whose vertices are the A_is. Join two A_is if they have a non-empty intersection. Is it true that the chromatic number of this graph is at most n?

To conclude this section I wish to call attention to a forthcoming book on extremal problems on set systems being written by Frankl, Füredi and Katona. I await the appearance of this book with great interest.

III. Combinatorial number theory

1. Let $a_1 < a_2 < \ldots < a_n$ be a B_2 sequence of Sidon, i.e. a sequence such that the sums $a_i + a_j$ are all distinct. Can this sequence be embedded into a perfect difference set? In other words, does there exist a sequence $b_1 < b_2 < \ldots < b_{p+1}$ such that all the differences are incongruent modulo p^2+p+1 and the a_is all occur among the b_js? (A slightly stronger requirement would be $a_i = b_i$, $1 \leqslant i \leqslant n$.) For the applications I have in mind it would suffice if there is an X and a B_2 sequence $1 \leqslant b_1 < \ldots < b_t \leqslant X$ such that $t = X^{1/2}(1+o(1))$ and all the a_is occur among the b_js. Unfortunately I could make no progress with these problems.

2. During the International Congress in Warsaw, Pisier asked me the following question. Call a sequence $a_1 < \ldots < a_n$ *independent* if the sums

$$\sum_{i=1}^{n} \varepsilon_i a_i, \qquad \varepsilon_i = 0 \quad \text{or} \quad 1$$

are all distinct. Now let $b_1 < b_2 < \ldots$ be an infinite sequence of integers and assume that there is a $\delta > 0$ such that for every n every subsequence of n terms of the b_is contains an independent subsequence of δn terms. Is it then true that our sequence $b_1 < b_2 < \ldots$ is the union of finitely many independent sequences? (Of course, their number should depend only on δ.) Pisier is not a combinatorialist but an outstanding analyst; he would need this lemma to characterize Sidon sets. As the reader can see "all roads lead to Rome". Unfortunately, so far I have been able to make no contribution to this very interesting question.

One of my oldest problems deals with independent sequences. Let $1 \leqslant a_1 < a_2 < \ldots < a_t \leqslant n$ be independent. Is it true that

$$\max t = \frac{\log n}{\log 2} + O(1)? \tag{21}$$

Leo Moser and I proved 30 years ago by using the second moment method that

$$\max t \leqslant \frac{\log n}{\log 2} + \frac{\log \log n}{2 \log 2} + O(1). \tag{22}$$

As far as I know, (22) is still the best known upper bound for max t. I offer a reward of US$500 for a proof or disproof of (21).

Here is another additive Pisier-type problem. Assume that the infinite sequence $B=\{b_1<b_2<\ldots\}$ has the property that for every n every subsequence of n terms of our sequence has a subsequence of δn terms which is a B_2 sequence. Is it then true that B is the union of C_δ B_2 sequences? Many generalizations and extensions are possible but I have no non-trivial results.

3. The problem posed by Pisier led me to several related questions. Let G be an infinite graph. Assume that there is a $\delta>0$ such that every set of e edges of G contains a subgraph of δe edges which do not contain a C_4. Is it then true that G is the union of C_δ graphs which do not contain a C_4? The answer is expected to be negative. C_4 could of course be replaced by any other graph H, but it is uncertain whether any non-trivial positive results can be obtained.

4. I conjectured that every sequence of integers $a_1<\ldots<a_n$ contains a subsequence $a_{i_1}<a_{i_2}<\ldots<a_{i_r}$ which is a B_2 sequence and $r=(1+o(1))n^{1/2}$. Komlós, Sulyok and Szemerédi proved this with $r<cn^{1/2}$.

I proved that there is a $B_2^{(2)}$ sequence $a_1<\ldots<a_n$ (i.e. the number of solutions of $a_i+a_j=t$ is greater than or equal to 2), such that the largest B_2 subsequence of it has at most $cn^{2/3}$ terms. Is this best possible? Or can $\frac{2}{3}$ be replaced by $\frac{1}{2}+\varepsilon$, or even by $\frac{1}{2}$?

J. Komlós, M. Sulyok and E. Szemerédi, Linear problems in combinatorial number theory. *Acta Math. Acad. Sci. Hungar.* **26** (1975), 113–121.

Clearly many further problems of the Pisier type can be asked, but it is not clear whether there ever are any non-trivial positive results.