MORE RESULTS ON SUBGRAPHS WITH MANY SHORT CYCLES

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1. Introduction

In a previous paper [2] the senior authors proved the following theorem: Let $G_1 = G_1(n;cn^2)$ be a graph of n vertices and cn^2 edges, c a positive constant. Then for n sufficiently large there is always a subgraph $G_2 = G_2(m;f(c)n^2)$ of G_1 every two edges of which lie together on a cycle of length at most 6 in G_2, and if two edges of G_2 have a common vertex they are on a cycle of length 4 in G_2. In that paper, we did not determine f(c) explicitly although the arguments used would yield something of the form $f(c) = \alpha c^3$. Here we replace c by $n^{-\varepsilon}$ for ε small and to some extent determine $f(n^{-\varepsilon})$. That is, we consider $G_1 = G_1(n;n^{2-\varepsilon})$ and try to find the subgraph G_2 having the largest number of edges with each pair of these edges on a short cycle. This problem leads to surprising and perhaps unexpected complications. We remark that in our previous paper we also discussed hypergraphs. In this paper, we do not do this but hope to return to that problem in the future (for the SENIOR author of course there may not be a future).

Here we prove the following theorem: For sufficiently large n each $G_1 = G_1(n;n^{2-\varepsilon})$ contains a $G_2 = G_2(m;cn^{2-3\varepsilon})$, where c does not depend on m, n, or ε, in which every two edges are on a cycle of length at most 6 in G_2, and that apart from the value of c this result is the best possible, i.e., 3ε cannot be replaced by any smaller value. If we further insist that any two edges of G_2 which have a common vertex are on a cycle of length 4 in G_2, then we can only show that there exists a $G_2 = G_2(m,cn^{2-5\varepsilon})$, but here we do not know if 5ε is best possible, and we cannot exclude the possibility that 5ε could be replaced by 3ε.

We further prove that there is a $G_2(m; cn^{2-2\varepsilon})$ every two edges of which are on a cycle of length at most 12. Here 2ε is of course best possible since our $G_1(n; n^{2-\varepsilon})$ could be the union of n^{ε} complete bipartite graphs each with $n^{2-2\varepsilon}$ edges, but 12 could perhaps be replaced by 8.

2. The Function $f_k(n,\varepsilon)$

We will always use $G_1 = G_1(n;\ell)$ to denote a graph having n vertices and ℓ edges and G_2 for a subgraph of G_1. By C_m we mean a cycle of length m. Let $f_k(n,\varepsilon)$ be the largest integer such that each $G_1 = G_1(n; n^{2-\varepsilon})$ for sufficiently large n contains a subgraph $G_2 = G_2(m; f_k(n,\varepsilon))$ each pair of edges of which lie together on a cycle of length at most 2k in G_2. As mentioned above, there exists a positive constant c such that $f_k(n,\varepsilon) \leq cn^{2-2\varepsilon}$.

It follows from Theorem 1** of [3] (or by the counting used in the proof of Theorem 1 in [2]) that for $0 < \varepsilon < 1/2$ and n sufficiently large there exists a positive constant c such that each $G_1 = G_1(n; n^{2-\varepsilon})$ contains $cn^{4-4\varepsilon}$ copies of C_4. In this case G_1 has an edge which is contained in at least $cn^{2-3\varepsilon}$ copies of C_4. A subgraph of G_1 whose edges are those of $cn^{2-3\varepsilon}$ C_4's all having a common edge has the property that each pair of its edges lie together on a cycle of length at most 6. Thus $f_3(n,\varepsilon) \geq cn^{2-3\varepsilon}$. Our first result states that, apart from the value of the constant, this bound is the best possible.

Theorem 1. For $0 < \varepsilon < 1/2$ there exist positive constants c_1 and c_2 such that $c_1 n^{2-3\varepsilon} \leq f_3(n,\varepsilon) \leq c_2 n^{2-3\varepsilon}$.

Proof. It remains only to establish the upper bound. For this we use a probabilistic argument to show that there exists $G_1 = G_1(n; n^{2-\varepsilon})$ which is such that any subgraph G_2 of G_1 in which each pair of edges are on a cycle of length at most 6 in G_2 has at most $c_2 n^{2-3\varepsilon}$ edges.

Let B be a complete bipartite graph with vertices $X_1, X_2, \ldots, X_\ell$ and $Y_1, Y_2, \ldots, Y_\ell$, $\ell = 2n^{1-\varepsilon}$, where each X_i is joined to every Y_j, $1 \leq i,j \leq \ell$. Also suppose that the $4n^{2-2\varepsilon}$ edges of B have been colored in some fashion with the colors $1, 2, \ldots, t$, $t = \frac{1}{4} n^{\varepsilon}$.

Now let $G_1 = G_1(n; n^{2-\varepsilon})$ be a bipartite graph with vertices $x(i,j)$, $y(i,j)$, $1 \leq i \leq \ell$, $1 \leq j \leq t$, where $x(i,j)$ is joined by an edge to $y(i',j')$ if and only if the edge joining X_i to $Y_{i'}$ in B has color $|j-j'|$ mod t. Each vertex of G_1 has valence $2n^{1-\varepsilon}$. Note also that if $j = j'$, then neither the pair of vertices $x(i,j)$ and $x(i,j')$ nor the pair $y(i,j)$ and $y(i,j')$ have a common neighbor in G_1 for any value of i, $1 \leq i \leq \ell$.

Now suppose that G_2 is a subgraph of G_1 in which each pair of edges lie together on a cycle of G_2 of length 4 or 6. For each i, $1 \leq i \leq \ell$, the vertex $x(i,j)$ is incident with an edge of G_2 for at most one value of j and the same is true for each $y(i,j)$ (this follows from the lack of common neighbors). Thus each choice of such a subgraph G_2 yields a coloring of (some of) the vertices of B with t colors in which X_i (Y_i) receives color j if $x(i,j)$ ($y(i,j)$) has valence at least one in G_2. Now $x(i,j)$ and $y(i',j')$ are joined by an edge in G_2 only if X_i has color j, $Y_{i'}$ has color j', and the edge $X_i Y_{i'}$ has color $|j-j'|$ mod t. For any choice of a vertex-coloring and an edge-coloring of B call an edge whose color is related to the colors of its endpoints in this fashion a "good" edge. We wish to estimate the number of these good edges when one of the colorings is selected at random.

For a fixed t-coloring of the vertices and a random t-coloring of the edges B the expected number of good edges is $(2n^{1-\varepsilon})^2 t^{-1} = 16n^{2-3\varepsilon}$. By an inequality of Chernoff (see Chapter 3 of [4]) the probability that such colorings result in at least $32n^{2-3\varepsilon}$ good edges is at most $\exp(-cn^{2-3\varepsilon})$, where c is a positive constant. Since there are $\exp(4\varepsilon n^{1-\varepsilon} \log \frac{1}{4} n)$ possible t-colorings of the vertices of B, the probability that for a given t-coloring of the edges of B there exists a t-coloring of the vertices which yields at least $32n^{2-3\varepsilon}$ good edges is at most $p = \exp[4\varepsilon n^{1-\varepsilon}(\log \frac{1}{4} n) - cn^{2-3\varepsilon}]$. For $0 < \varepsilon < 1/2$ and n sufficiently large we have $p < 1$. It follows that there exists a constant c_2 and a t-coloring of the edges of B which yields at most $c_2n^{2-3\varepsilon}$ good edges for any t-coloring of the vertices of B. Hence there is a graph $G_1 = G_1(n;n^{2-\varepsilon})$ such that each subgraph G_2 of G_1 having the required short cycles has at most $c_2n^{2-3\varepsilon}$ edges.

Although $c_1n^{2-3\varepsilon} \le f_3(n,\varepsilon) \le c_2n^{2-3\varepsilon}$, our next result shows that for $k \ge 6$ the bound $f_k(n,\varepsilon) \le cn^{2-2\varepsilon}$ is essentially the correct one. (Perhaps this remains so for a smaller value of $k > 3$.)

Theorem 2. There exists a positive constant c such that $f_6(n,\varepsilon) \ge cn^{2-2\varepsilon}$.

Proof. By standard results (see [1]) we may assume that our graph $G_1 = G_1(n;n^{2-\varepsilon})$ is bipartite and that each vertex has valence at least $n^{1-\varepsilon}$. In this case there exist two vertices x_1 and x_2 and vertices $y_1,y_2,\ldots,y_\ell$, $\ell = cn^{1-2\varepsilon}$, c a positive constant, such that each y_i, $1 \le i \le \ell$, is joined to both x_1 and x_2. Let $z_1,z_2,\ldots,z_t$ be the set of all vertices of G_1 which are joined to at least one of the y's. Since each y_i has valence at least $n^{1-\varepsilon}$, we have $t \ge n^{1-\varepsilon}$. Now let $w_1,\ldots,w_s$ denote the set of all vertices (other than the y's) which are joined to at least two of the z's. The subgraph of G_1 spanned by x_1,x_2, and all of the y's, z's, and w's is our desired subgraph G_2. The number of edges joining the w's and the z's is at least $c'n^{2-2\varepsilon}$ for a positive constant c' since by omitting those w's which are joined to only one of the z's we lose at most n edges. Thus G_2 has at least $c'n^{2-2\varepsilon}$ edges. To check that each pair of edges of G_2 is on a cycle of length at most 12 a number of cases must be considered. We note only that edges $y_iz_{i'}$ and $y_jz_{j'}$, $i \ne j$, $i' \ne j'$, are on a cycle of length 12 of the form $y_i, z_{i'}, w_a, z_b, y_c, x_1, y_d, z_e, w_f, z_{j'}, y_j, x_2, y_i$, while for many pairs of edges shorter cycles exist.

It may be that Theorem 2 remains true for $k = 5$ or even for $k = 4$. About this we know only that $f_5(n,\varepsilon) \le cn^{2-5\varepsilon/2}$ which can be seen by letting xy be an edge of $G_1(n;cn^{2-\varepsilon})$ which is contained in at least $cn^{2-3\varepsilon}$ copies of C_4 (as for

Theorem 1) and taking as G_2 the subgraph spanned by the vertices of these C_4's as well as all of the vertices which are adjacent to at least two vertices each a neighbor of x on one of the C_4's.

In the earlier paper concerning $G(n;cn^2)$ we found a subgraph $G_2 = G_2(m;f(c)n^2)$ with the additional property that each pair of edges having a common vertex are on a cycle of length 4 in G_2. Here we have only the following result.

Theorem 3. Given ε, $0 < \varepsilon < 1/2$, and n sufficiently large, there exists a positive constant c such that each $G_1 = G_1(n;n^{2-\varepsilon})$ contains a subgraph G_2 with $cn^{2-5\varepsilon}$ edges with the property that each pair of edges of G_2 are on a cycle of length at most 6 in G_2 and any two of these edges with a common vertex are on a C_4 in G_2.

Proof. As for Theorem 2 we may assume that G_1 is bipartite with each vertex of valence at least $n^{1-\varepsilon}$, and that there exist vertices $x_1,x_2,y_1,\ldots,y_\ell$, $\ell = n^{1-2\varepsilon}$, where each y_i, $1 \leq i \leq \ell$, is joined to both x_1 and x_2. Let the vertices joined to one or more of the y's be $z_1,z_2,\ldots,z_t$ $(t \leq n)$. Since each y_i has valence $n^{1-\varepsilon}$ there are $n^{2-3\varepsilon}$ edges joining the y's and z's and (by omitting some of the z's) we may assume that each z is joined to at least $n^{1-3\varepsilon}$ y's. Thus there exist constants c_1 and c_2 such that there are $c_1n^{2-4\varepsilon}$ pairs of y's and each z is joined to at least $c_2n^{2-6\varepsilon}$ of these pairs. It follows that there exist two of the y's, say y_1 and y_2, and vertices $z_1,z_2,\ldots,z_s$, $s = c_2n^{1-2\varepsilon}$, among the z's, such that z_i is joined to both y_1 and y_2 for each i, $1 \leq i \leq s$. Now x_1 and x_2 are each joined to all of the y's, while y_1 and y_2 are each joined to every z_i for $1 \leq i \leq s$, as well as to x_1 and x_2. The number of edges joining the y's and the z's is $c_3n^{2-5\varepsilon}$ and every pair of edges joining y's to vertices among $x_1,x_2,z_1,\ldots,$ and z_t is on a cycle of length at most 6 with any pair of edges which have a vertex in common being on a cycle of length 4.

3. Further Problems

Questions remain concerning the exact values of the constants in all of our theorems and of course about the correct bounds for $f_4(n,\varepsilon)$ and $f_5(n,\varepsilon)$. It would be more interesting however to determine whether Theorem 3 can be improved, and in particular whether 5ε can be replaced by 3ε in that result. More generally is it true that the value of $f_k(n,\varepsilon)$ remains unchanged for all $k \geq 3$ (except possibly for the constant c) if we insist that a pair of edges of the subgraph having a common vertex be on a cycle of the subgraph of length at most 2k-2?

References

[1] B. Bollobás, "Extremal Graph Theory," Academic Press, New York, 1978.

[2] R. A. Duke and P. Erdös, Subgraphs in which each pair of edges lie on a short cycle, Proc. 13th S.E. Conf. Comb. Graph Theory, and Computing, Congressus Numerantium, 1982, Vol. 35, pp. 253-260.

[3] P. Erdös and M. Simonovitz, Oversaturated graphs and hypergraphs, Combinatorica, Vol. 3, 1983, 181-192.

[4] P. Erdös and J. Spencer, "Probabilistic Methods in Combinatorics," Academic Press, New York, 1974.