Studies in Pure Mathematics
To the Memory of Paul Turán

Some asymptotic formulas on generalized divisor functions I

by

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1. Throughout this paper, we use the following notations:

$c, c_1, c_2, \dots, X_0, X_1, \dots$ denote positive absolute constants. We denote the number of the elements of the finite set S by $|S|$. We write $e^x = \exp(x)$. $\nu(n)$ denotes the number of the distinct prime factors of n. We denote the least prime factor of n by $p(n)$, while the greatest prime factor of n is denoted by $P(n)$.

Let A be a finite or infinite sequence of positive integers $a_1 < a_2 < \dots$. Then we write

$$N_A(x) = \sum_{\substack{a \in A \\ a \leqq x}} 1,$$

$$f_A(x) = \sum_{\substack{a \in A \\ a \leqq x}} \frac{1}{a},$$

$$d_A(n) = \sum_{\substack{a \in A \\ a/n}} 1$$

(in other words, $d_A(n)$ denotes the number of divisors amongst the a_i's) and

$$D_A(x) = \max_{1 \leqq n \leqq x} d_A(x).$$

The aim of this paper is to investigate the function $D_A(x)$. Clearly

(1) $$\sum_{1 \leqq n \leqq x} d_A(n) = x f_A(x) + O(x).$$

One would expect that if $N_A(x) \to +\infty$ then also

(2) $$\lim_{x \to +\infty} \frac{D_A(x)}{f_A(x)} = +\infty.$$

(2) is trivial if $f_A(x) < C$ thus we can assume

$$f_A(x) \to +\infty. \tag{3}$$

The special case when

$$(a_i, a_j) = 1 \quad \text{for all} \quad 1 \leqq i < j \tag{4}$$

was posed as a problem in [2]. Furthermore, we guessed there that condition (4) can be dropped, in other words, (2) holds for all infinite sequences. To our great surprise, we disproved (2); Section 2 will be devoted to the counter-example. On the other hand, we prove in Section 3 that $\lim_{x \to +\infty} \inf N_A(x) \left(\frac{x \log \log x}{\log x} \right)^{-1} > c_1$ implies (2). We believe that also the weaker condition $f_A(x) (\log \log x)^{-1} \to +\infty$ implies (2). We hope to return to this question in a subsequent paper.

Furthermore, we prove in Section 3 that (3) implies that

$$\lim_{x \to +\infty} \sup \frac{D_A(x)}{f_A(x)} = +\infty. \tag{5}$$

Perhaps

$$\lim_{x \to +\infty} \sup D_A(x) / f_A(x)^{(1-\varepsilon)\log f_A(x)} = +\infty$$

also holds; we will return to this problem in Part II of this paper. In Section 3, we prove several other theorems concerning various sharpenings of (2) and (5).

Theorem 1. *There exist positive constants c_2, c_3 and an infinite sequence A of positive integers such that for an infinite sequence $x_1 < x_2 < \ldots < x_k < \ldots$ of positive integers we have*

$$f_A(x_k) > c_2 \log \log x_k \tag{6}$$

and

$$\frac{D_A(x_k)}{f_A(x_k)} < c_3. \tag{7}$$

Proof. We are going to construct finite sequences satisfying inequalities corresponding to (6) and (7) at first.

By a theorem of HARDY and RAMANUJAN [5], there exist positive constants δ and X_1 such that if $x > X_1$ then uniformly for all $\sqrt{x} \leqq y \leqq x$, the conditions $b \leqq y$ and $v(b) < 2 \log \log x$ hold for all but $\frac{y}{(\log x)^\delta}$ integers b. (See also [1].)

For any positive integer $x \geqq 10$ and for $1 \leqq j \leqq (\log x)^{\delta/2}$, let $B_j(x)$ denote the set of those integers b for which

(i) $\frac{x}{2^j} < b \leqq \frac{x}{2^{j-1}}$,

(ii) $p(b) > 2^j$,

(iii) $\mu(b) \neq 0$

and

(iv) $\nu(b) < 2 \log\log x$

hold and let

$$B(x) = \bigcup_{1 \leqq j \leqq (\log x)^{\delta/2}} B_j(x).$$

We will show that there exist constants X_2 and c_4 such that for $x \geqq X_2$, we have

$$\sum_{b \in B(x)} \frac{1}{b} > c_4 \log\log x \tag{8}$$

and

$$D_{B(x)}(x) < 2 \log\log x. \tag{9}$$

By using standard methods of the prime number theory (see e.g. [3] or [4]), it can be shown easily that there exist constants c_5 and X_3 such that if $x > X_3$ then uniformly for all y and z for which $\sqrt{x} < y$ and $z \leqq 2^{(\log x)^{\delta/2}}$, the number of the integers b satisfying the conditions $y \leqq b \leqq 2y$, $p(b) > z$ and $\mu(b) \neq 0$ is greater than

$$c_5 y \prod_{p \leqq z}\left(1 - \frac{1}{p}\right) \prod_{p > z}\left(1 - \frac{1}{p^2}\right) > c_6 \frac{y}{\log z}.$$

Thus for $x > X_3$, the number of the integers b satisfying (i), (ii) and (iii) (for fixed j) is greater than

$$c_6 \frac{x/2^j}{\log 2^j} = c_7 \frac{x}{j2^j}$$

uniformly for $1 \leqq j \leqq (\log x)^{\delta/2}$.

On the other hand, by

$$\frac{x}{2^j} \geqq \frac{x}{2^{(\log x)^{\delta/2}}} > \frac{x}{\sqrt{x}} = \sqrt{x},$$

the definition of δ yields that for $x>X_1$, (iv) holds for all but

$$\frac{x/2^{j-1}}{(\log x)^{\delta}}=\frac{x}{2^{j-1}(\log x)^{\delta}}$$

of the integers b satisfying (i).

Thus for $x \geqq X_4$, we have

$$|B_j(x)|>c_7\frac{x}{j2^j}-\frac{x}{2^{j-1}(\log x)^{\delta}}=c_7\frac{x}{j2^j}\left(1-\frac{2}{c_7}\frac{j}{(\log x)^{\delta}}\right)>c_8\frac{x}{j2^j}$$

for all $1\leqq j\leqq(\log x)^{\delta/2}$, hence

$$\sum_{b\in B(x)}\frac{1}{b}=\sum_{1\leqq j\leqq(\log x)^{\delta/2}}\sum_{b\in B_j(x)}\frac{1}{b}\geqq\sum_{1\leqq j\leqq(\log x)^{\delta/2}}\sum_{b\in B_j(x)}\frac{1}{x/2^{j-1}}=$$

$$=\sum_{1\leqq j\leqq(\log x)^{\delta/2}}|B_j(x)|\frac{2^{j-1}}{x}>\sum_{1\leqq j\leqq(\log x)^{\delta/2}}c_8\frac{1}{2j}>c_9\log(\log x)^{\delta/2}>c_{10}\log\log x$$

for $x>X_5$ which proves (8).

In order to prove (9), note that if

$$b_1u=b_2v\leqq x$$

for some positive integers $b_1\in B(x)$, $b_2\in B(x)$, u, v, and $b_1<b_2$ then by the construction of the set $B(x)$, we have

$$p(b_1)>\frac{x}{b_1}\geqq u=\frac{b_2}{b_1}v>v$$

thus $(b_1, v)=1$ and $b_1=\frac{b_2v}{u}\Big/ b_2v$, hence b_1/b_2. Thus if $n\leqq x$, and $b_1<b_2<\ldots<b_r$ denote all the positive integers b_i such that $b_i\in B(x)$ and b_i/n then

(10) $$b_1/b_2/\ldots/b_r$$

must hold. By the construction of the set $B(x)$, we have

(11) $$\mu(b_r)\neq 0$$

and

(12) $$\nu(b_r)<2\log\log x\,.$$

(10) and (11) imply that

$$\nu(b_1)<\nu(b_2)<\ldots<\nu(b_r)$$

thus with respect to (12),

$$d_{B(x)}(n)=r\leqq\nu(b_r)<2\log\log x$$

for all $n\leqq x$ which proves (9).

Finally, let $x_1=\max\{10,\ [X_2]+1\}$ and $x_k=[\exp\{\exp(\exp x_{k-1})\}]+1$ for $k=2, 3, \dots$, and let

$$A=\bigcup_{k=1}^{+\infty} B(x_k).$$

Then by (8), we have

$$f_A(x_k)=\sum_{\substack{a\in A\\ a\leqq x_k}}\frac{1}{a}\geqq\sum_{a\in B(x_k)}\frac{1}{a}>c_4\log\log x_k \tag{13}$$

for $k=1, 2, \dots$ which proves (6).

Furthermore, (9) yields that for $k=2, 3, \dots$ and $n\leqq x_k$, we have

$$d_A(n)\leqq\sum_{i=1}^{k}d_{B(x_i)}(n)=\sum_{i=1}^{k-1}d_{B(x_i)}(n)+d_{B(x_k)}(n)\leqq$$

$$\leqq\sum_{i=1}^{k-1}\sum_{b\in B(x_i)}1+D_{B(x_k)}(x_k)<\sum_{b\leqq x_{k-1}}1+2\log\log x_k=$$

$$=x_{k-1}+2\log\log x_k<\log\log\log x_k+2\log\log x_k<3\log\log x_k$$

hence

$$D_A(x_k)<3\log\log x_k. \tag{14}$$

(13) and (14) yield (7) and the proof of Theorem 1 is completed.

We note that we could sharpen Theorem 1 in the following way:

Theorem 1'. *There exists an infinite sequence A of positive integers such that for an infinite sequence $x_1<x_2<\dots<x_k<\dots$ of positive integers we have*

$$\liminf_{k\to+\infty}\frac{f_A(x_k)}{e^{-\gamma}\log\log x_k}=1 \tag{6'}$$

and

$$\limsup_{k\to+\infty}\frac{D_A(x_k)}{f_A(x_k)}=1 \tag{7'}$$

where γ denotes the Euler-constant.

Note that (7') is best possible as (1) shows.

In fact, Theorem 1' could be proved by the following construction:

Let x_1 be a large number, and for $k=2, 3, \ldots$, let x_k be sufficiently large in terms of k and x_{k-1}. For $k=1, 2, \ldots$, let $B(x_k)$ denote the set of those integers b for which

(i) $x_k^{1/2} < b < x_k$,

(ii) $p(b) > \frac{x_k}{b}$,

(iii) $\mu(b) \neq 0$,

(iv) $\nu(b) < \left(1+\frac{1}{k}\right) \log\log x_k$,

(v) if the prime factors of b are $p_1 < p_2 < \ldots < p_{\nu(b)}$ then $p_{i+1} > p_1 p_2 \ldots p_i$ holds for less than $\left(1+\frac{2}{k}\right) e^{-\gamma} \log\log x_k$ of the integers $1 \leqq i \leqq \nu(b)$.

Finally, let

$$A = \bigcup_{k=1}^{+\infty} B(x_k).$$

It can be shown easily that for this sequence A, we have

$$\limsup_{k\to+\infty} \frac{D_A(x_k)}{e^{-\gamma} \log\log x_k} \leqq 1. \tag{15}$$

Combining the methods of probability theory with Brun's sieve (see e.g. [3] or [4]) it can be proved that also (6′) holds. However, this proof would be very complicated; this is the reason of that that we have worked out the weaker version discussed in Theorem 1. (1), (6′) and (15) yield also (7′).

Theorem 2. *If*

$$\lim_{x\to+\infty} f_A(x) = +\infty \tag{16}$$

then we have

$$\limsup_{x\to+\infty} D_A(x) \left(\frac{\log x}{\log\log x}\right)^{-1} \geqq 1. \tag{17}$$

Note that this theorem is best possible as the sequence A consisting of all the prime number shows.

Proof. We are going to show at first that (16) implies that for all $\varepsilon > 0$, there exist infinitely many integers y such that

$$N_A(y) > \frac{y}{(\log y)^{1+\varepsilon}}. \tag{18}$$

In fact, let us assume indirectly that for some $\varepsilon>0$ and $y>y_0(\varepsilon)$ we have

$$N_A(y)\leqq \frac{y}{(\log y)^{1+\varepsilon}}.$$

Then partial summation yields that for $x\to+\infty$ we have

$$f_A(x)=\sum_{a\leqq x}\frac{1}{a}=\sum_{y=1}^{x}\frac{N_A(y)-N_A(y-1)}{y}=\sum_{y=1}^{x}N_A(y)\left(\frac{1}{y}-\frac{1}{y+1}\right)+\frac{N_A(x)}{x+1}=$$

$$=\sum_{y=1}^{x}\frac{N_A(y)}{y(y+1)}+\frac{N_A(x)}{x+1}=O\left(\sum_{y=1}^{x}\frac{y/(\log y)^{1+\varepsilon}}{y^2}\right)+O\left(\frac{x/(\log x)^{1+\varepsilon}}{x}\right)=$$

$$=O\left(\sum_{y=1}^{x}\frac{1}{y(\log y)^{1+\varepsilon}}\right)+O\left(\frac{1}{(\log x)^{1+\varepsilon}}\right)=O(1)$$

in contradiction with (16) and this contradiction proves the existence of infinitely many integers y satisfying (18) (for all $\varepsilon>0$).

Let us fix some $\varepsilon>0$ and let y be a large integer satisfying (18). Put

$$X=\prod_{\substack{a\in A\\ a\leqq y}}a.$$

Then

$$X\leqq\prod_{\substack{a\in A\\ a\leqq y}}y=y^{N_A(y)}$$

hence

$$\log X\leqq N_A(y)\log y, \tag{19}$$

and for large y, we have

$$\log X=\sum_{\substack{a\in A\\ a\leqq y}}\log a\geqq\sum_{\substack{a\in A\\ 3\leqq a\leqq y}}\log a>$$

$$>\sum_{\substack{a\in A\\ 3\leqq a\leqq y}}\log 3=(N_A(y)-N_A(2))\log 3\geqq(N_A(y)-2)\log 3>N_A(y)$$

thus by (18),

$$\log\log X>\log N_A(y)>\log\frac{y}{(\log y)^{1+\varepsilon}}>(1-\varepsilon)\log y \tag{20}$$

for sufficiently large y.

(19) and (20) yield that

$$N_A(y) \geqq \frac{\log X}{\log y} > \frac{\log X}{\frac{1}{1-\varepsilon}\log\log X} = (1-\varepsilon)\frac{\log X}{\log\log X}. \tag{21}$$

Furthermore, we have

$$D_A(X) \geqq d_A(X) = \sum_{\substack{a\in A\\ a|X}} 1 \geqq N_A(y) \tag{22}$$

since $X = \prod_{\substack{a\in A\\ a\leqq y}} a$ is divisible by all the $N_A(y)$ integers a satisfying $a\in A$, $a \leqq y$.

(21) and (22) yield that

$$D_A(X) > (1-\varepsilon)\frac{\log X}{\log\log X}.$$

For all $\varepsilon > 0$, this holds for infinitely many integers X and this proves (17).

Theorem 3. *If* $x > X_0$ *and*

$$N_A(x) > 5\frac{x\log\log x}{\log x} \tag{23}$$

then there exists a positive integer X *such that*

$$\frac{x}{\log x} < X < \exp(x) \tag{24}$$

and

$$\frac{d_A(X)}{\log X} > \exp\left(\frac{1}{20}\frac{\log x}{x}N_A(x)\right). \tag{25}$$

Note that by (23) and (24), the right-hand side of (25) is

$$\exp\left(\frac{1}{5}\frac{\log x}{x}N_A(x)\right) > \exp(\log\log x) = \log x > \log\log X \to +\infty$$

as $x \to +\infty$.

Theorem 4. *If* A *is an infinite sequence such that*

$$\liminf_{x\to+\infty} N_A(x)\left(\frac{x\log\log x}{\log x}\right)^{-1} > 5 \tag{26}$$

then we have

$$\lim_{x\to+\infty} \frac{D_A(x)}{\log x} = +\infty. \tag{27}$$

Note that for large x, we have

$$f_A(x) = \sum_{\substack{a\in A\\ a\leqq x}} \frac{1}{a} \leqq \sum_{a\leqq x} \frac{1}{a} < 2\log x \tag{28}$$

thus (25) implies that also

$$\lim_{x\to+\infty} \frac{D_A(x)}{f_A(x)} = +\infty$$

holds.

We are going to prove Theorems 3 and 4 simultaneously.

Proof of Theorems 3 and 4. Assume that $x > X_0$ and for a finite or infinite sequence A, we have

$$N_A(x) > 5\frac{x\log\log x}{\log x}. \tag{29}$$

Let t be a real number such that

$$\frac{5}{4}\log\log x \leqq \log t \leqq \frac{1}{4}\frac{\log x}{x} N_A(x). \tag{30}$$

Then obviously, we have

$$\log t \leqq \frac{1}{4}\frac{\log x}{x} x = \frac{1}{4}\log x$$

hence

$$t \leqq x^{1/4}. \tag{31}$$

Let A^* denote the set of those integers a for which $a\in A$, $a\leqq x$ and $P(a) > \frac{x}{t}$ hold. It is well known that

$$\sum_{p\leqq y} \frac{1}{p} = \log\log y + c_{11} + O\left(\frac{1}{\log y}\right). \tag{32}$$

(30), (31) and (32) yield that

$$\sum_{\substack{1\leqq n\leqq x\\ P(n)>x/t}} 1 \leqq \sum_{x/t<p\leqq x}\ \sum_{\substack{1\leqq n\leqq x\\ p/n}} 1 =$$

$$= \sum_{x/t<p\leqq x}\left[\frac{x}{p}\right] \leqq \sum_{x/t<p\leqq x}\frac{x}{p} = x\left(\sum_{p\leqq x}\frac{1}{p} - \sum_{p\leqq x/t}\frac{1}{p}\right) =$$

(33)

$$=x\left\{\left(\log\log x+c_{11}+O\left(\frac{1}{\log x}\right)\right)-\left(\log\log x/t+c_{11}+O\left(\frac{1}{\log x/t}\right)\right)\right\}=$$

$$=-x\log\left(1-\frac{\log t}{\log x}\right)+O\left(\frac{x}{\log x}\right)<2x\frac{\log t}{\log x}+O\left(\frac{x}{\log x}\right)<3x\frac{\log t}{\log x}$$

since

$$-\log(1-y)=\sum_{k=1}^{+\infty}\frac{y^k}{k}<\sum_{k=1}^{+\infty}y^k=\frac{y}{1-y}<2y \quad \text{for} \quad 0<y<\frac{1}{2},$$

and

$$0<\frac{\log t}{\log x}<\frac{1}{4}$$

by (30) and (31).

(30) and (33) yield that

$$|A^*|\geqq N_A(x)-\sum_{\substack{1\leqq n\leqq x\\ P(n)>x/t}}1=N_A(x)\left(1-\frac{1}{N_A(x)}\sum_{\substack{1\leqq n\leqq x\\ P(n)>x/t}}1\right)=$$

(34)

$$=N_A(x)\left(1-\frac{\log x}{4x\log t}\sum_{\substack{1\leqq n\leqq x\\ P(n)>x/t}}1\right)>N_A(x)\left(1-\frac{\log x}{4x\log t}\cdot 3x\frac{\log t}{\log x}\right)=\frac{1}{4}N_A(x).$$

Let us denote the least common multiple of the elements of A^* by X. Then with respect to (34), we have

$$d_A(X)\geqq d_{A^*}(X)=|A^*|>\frac{1}{4}N_A(x). \tag{35}$$

Furthermore, if $a\in A^*$ then $a\leqq x$ and $P(a)\leqq x/t$ thus we have

$$a\Big/\prod_{p\leqq x/t}p^{[\log x/\log p]}$$

hence

$$X\Big/\prod_{p\leqq x/t}p^{[\log x/\log p]}$$

which implies that

$$X\leqq\prod_{p\leqq x/t}p^{[\log x/\log p]}\leqq p\prod_{p\leqq x/t}x=x^{\pi(x/t)}. \tag{36}$$

Using the prime number theorem or a more elementary theorem, we obtain from (36) with respect to (31) that

$$\log X \leqq \pi(x/t) \log x < 2 \frac{x/t}{\log x/t} \log x \leqq \tag{37}$$

$$\leqq 2 \frac{x}{t \log (x/x^{1/4})} \log x = \frac{8}{3} \frac{x}{t}.$$

In order to deduce Theorem 3 from the construction above, assume that A satisfies the conditions in Theorem 3, and put

$$\log t = \frac{1}{4} \frac{\log x}{x} N_A(x). \tag{38}$$

Then by (23), we have

$$\log t > \frac{1}{4} \frac{\log x}{x} \cdot 5 \frac{x \log \log x}{\log x} = \frac{5}{4} \log \log x, \tag{39}$$

while the second inequality in (30) holds by the definition of t. Thus by (23), (35), (37) and (38), the construction above yields the existence of an integer X such that

$$\frac{d_A(X)}{\log X} > \frac{N_A(x)/4}{8x/3t} = \frac{3}{32} \cdot \frac{N_A(x)t}{x} = \frac{3}{32} \cdot \frac{N_A(x)}{x} \exp\left(\frac{1}{4} \cdot \frac{\log x}{x} N_A(x)\right) =$$

$$= \exp\left(\frac{1}{4} \cdot \frac{\log x}{x} N_A(x) + \log \frac{3}{32} \cdot \frac{N_A(x)}{x}\right) >$$

$$> \exp\left(\frac{1}{4} \cdot \frac{\log x}{x} N_A(x) + \log \frac{3}{32} \cdot \frac{5 \log \log x}{\log x}\right) >$$

$$> \exp\left(\frac{1}{4} \cdot \frac{\log x}{x} N_A(x) - \log \log x\right) >$$

$$> \exp\left\{\frac{1}{4} \cdot \frac{\log x}{x} N_A(x) \left(1 - 4 \cdot \frac{1}{N_A(x)} \cdot \frac{x \log \log x}{\log x}\right)\right\} >$$

$$> \exp\left\{\frac{1}{4} \cdot \frac{\log x}{x} N_A(x) \left(1 - \frac{4}{5}\right)\right\} = \exp\left(\frac{1}{20} \cdot \frac{\log x}{x} N_A(x)\right).$$

Finally, by the definition of X and with respect to (23) and (34), we have

$$X \geqq \max_{a \in A^*} a \geqq |A^*| > \frac{1}{4} N_A(x) > \frac{1}{4} \cdot 5 \frac{x \log \log x}{\log x} > \frac{x}{\log x},$$

while (36) and (39) yield that

$$X < \exp\left(\frac{8}{3}\cdot\frac{x}{t}\right) < \exp\left(\frac{8}{3}\cdot\frac{x}{(\log x)^{5/4}}\right) < \exp(x)$$

and this completes the proof of Theorem 3.

In order to prove Theorem 4, assume that an infinite sequence A satisfies (26) and let y be a large number; we are going to show by using the construction above that $\frac{D_A(y)}{\log y}$ is large. Define x by

$$x = \frac{1}{3}\log y(\log\log y)^{5/4}$$

and put $t = (\log x)^{5/4}$. Then for sufficiently large y, (29) holds by (26). Furthermore,

$$\frac{1}{4}\frac{\log x}{x}N_A(x) > \frac{1}{4}\frac{\log x}{x}\cdot 5\frac{x\log\log x}{\log x} = \frac{5}{4}\log\log x = \log t$$

thus also (30) holds. The construction above yields the existence of an integer X such that (35) and (37) hold. We obtain from (37) that

$$X < \exp\left(\frac{3}{8}\frac{x}{t}\right) = \exp\left\{\frac{8}{3}\cdot\frac{1}{3}\frac{\log y(\log\log y)^{5/4}}{\left(\log\left(\frac{1}{3}\log y(\log\log y)^{5/4}\right)\right)^{5/4}}\right\} <$$

$$< \exp\left(\frac{8}{9}\frac{\log y\,(\log\log y)^{5/4}}{(\log\log y)^{5/4}}\right) = y^{8/9} < y,$$

thus with respect to (29) and (35), we have

$$\frac{D_A(y)}{\log y} \geqq \frac{d_A(X)}{\log y} > \frac{N_A(x)/4}{\log y} > \frac{4}{5}\frac{x\log\log x}{\log x\log y} =$$

$$= \frac{4}{5}\frac{\frac{1}{3}\log y\,(\log\log y)^{5/4}\log\log\left(\frac{1}{3}\log y\,(\log\log y)^{5/4}\right)}{\log\left(\frac{1}{3}\log y\,(\log\log y)^{5/4}\right)\log y} > \tag{40}$$

$$> \frac{4}{15}\frac{(\log\log y)^{5/4}\log\log\log y}{2\log\log y} > \frac{2}{15}(\log\log y)^{1/4}\log\log\log y$$

which completes the proof of Theorem 4.

Theorems 3 and 4 are best possible (except the values of the constants on the right hand sides of (23) and (26), respectively) as the following theorem shows:

Theorem 5. *There exists an infinite sequence A of positive integers such that*

$$\lim_{x\to+\infty} \inf N_A(x)\left(\frac{x\log\log x}{\log x}\right)^{-1} \geqq 1 \tag{41}$$

and

$$d_A(x) \leqq \log x \tag{42}$$

for all x.

Proof. Let A consist of all the integers a of the form $a=pk$ where p is a prime number and $1\leqq k\leqq \log p$. Then by the prime number theorem (or a more elementary theorem) and (32) we have

$$\sum_{\substack{a\in A\\ a\leqq x}} 1 = \sum_{p\leqq x}\ \sum_{1\leqq k\leqq \min\{\log p,\, x/p\}} 1 \geqq \sum_{\frac{x}{\log x-2\log\log x}<p\leqq x}\ \sum_{1\leqq k\leqq \frac{x}{p}} 1 \geqq$$

$$\geqq \sum_{\frac{x}{\log x-2\log\log x}<p\leqq x}\left(\frac{x}{p}-1\right) = x\left(\sum_{p\leqq x}\frac{1}{p} - \sum_{p\leqq \frac{x}{\log x-2\log\log x}}\frac{1}{p}\right) + O\left(\frac{x}{\log x}\right) =$$

$$= x\left(\log\log x - \log\log\frac{x}{\log x-2\log\log x} + O\left(\frac{1}{\log x}\right)\right) + O\left(\frac{x}{\log x}\right) =$$

$$= -x\log\left(1-\frac{\log(\log x-2\log\log x}{\log x}\right. + O\left(\frac{x}{\log x}\right) =$$

$$=(1+o(1))\frac{x\log(\log x-2\log\log x)}{\log x} + O\left(\frac{x}{\log x}\right) = (1+o(1))\frac{x\log\log x}{\log x}$$

which proves (41).

Let $x\geqq 2$ be an integer and let $x=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$ where $p_1<p_2<\dots<p_r$ are prime numbers and $\alpha_1, \alpha_2, \dots, \alpha_r$ are positive integers. For $i=1, 2, \dots, r$, let S_i denote the set of the integers a for which $a\in A$, a/x and $P(a)=p_i$ hold.

By the definition of the set A, $a\in S_i$ implies that a can be written in the form $a=p_ik$ where $1\leqq k\leqq \log p_i$. Thus obviously, we have

$$|S_i| \leqq \sum_{1\leqq k\leqq \log p_i} 1 \leqq \log p_i$$

hence

$$d_A(x) = \sum_{\substack{a \in A \\ a/x}} 1 = \sum_{i=1}^{r} \sum_{\substack{a \in A \\ a/x \\ P(a)=p_i}} 1 = \sum_{i=1}^{r} |S_i| \leqq \sum_{i=1}^{r} \log p_i =$$

$$= \log\left(\prod_{i=1}^{r} p_i\right) \leqq \log\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leqq \log x$$

and this completes the proof of Theorem 5.

Theorems 2 and 3 imply that

Theorem 6. *If*

$$\lim_{x \to +\infty} f_A(x) = +\infty$$

then we have

$$\limsup_{x \to +\infty} \frac{D_A(x)}{f_A(x)} = +\infty. \tag{43}$$

Proof. Assume at first that

$$f_A(x) = o\left(\frac{\log x}{\log\log x}\right). \tag{44}$$

We have

$$\frac{D_A(x)}{f_A(x)} = \frac{D_A(x)}{\frac{\log x}{\log\log x}} \cdot \frac{\frac{\log x}{\log\log x}}{f_A(x)}.$$

Here the first factor is $\geqq \frac{1}{2}$ for infinitely many integers x by Theorem 2, while the second factor tends to $+\infty$ by (44) which implies (43).

Assume now that

$$\limsup_{x \to +\infty} \frac{f_A(x)}{\frac{\log x}{\log\log x}} > 0. \tag{45}$$

We are going to show that this implies that there exist infinitely many integers x satisfying

$$N_A(x) > 5\frac{x \log\log x}{\log x}. \tag{46}$$

Assume indirectly that for $x > X_0$ we have

$$N_A(x) \leqq 5\frac{x \log\log x}{\log x}.$$

Then partial summation yields that

$$f_A(x)=\sum_{\substack{a\in A\\ a\leqq x}}\frac{1}{a}=\sum_{y\leqq x}\frac{N_A(y)-N_A(y-1)}{y}=\sum_{y\leqq x}N_A(y)\left(\frac{1}{y}-\frac{1}{y+1}\right)+\frac{N_A(x)}{x+1}=$$

$$=\sum_{y\leqq x}\frac{N_A(y)}{y(y+1)}+\frac{N_A(x)}{x+1}\leqq\sum_{y\leqq x}\frac{N_A(y)}{y^2}+\frac{N_A(x)}{x}=$$

$$=O\left(\sum_{y\leqq x}\frac{\log\log y}{y\log y}\right)+O\left(\frac{\log\log x}{\log x}\right)=O((\log\log x)^2)$$

in contradiction with (45) which proves the existence of infinitely many integers satisfying (46). By Theorem 3, this implies that

$$\limsup_{x\to+\infty}\frac{D_A(x)}{\log x}=+\infty. \tag{47}$$

Obviously, we have

$$f_A(x)=\sum_{\substack{a\in A\\ a\leqq x}}\frac{1}{a}\leqq\sum_{a\leqq x}\frac{1}{a}\sim\log x$$

thus

$$\liminf_{x\to+\infty}\frac{\log x}{f_A(x)}\geqq 1. \tag{48}$$

(47) and (48) yield that

$$\limsup_{x\to+\infty}\frac{D_A(x)}{f_A(x)}=\limsup_{x\to+\infty}\frac{D_A(x)}{\log x}\cdot\frac{\log x}{f_A(x)}=+\infty$$

and this completes the proof of Theorem 6.

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