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FINITE LINEAR SPACES AND PROJECTIVE PLANES

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In 1948, De Bruijn and Erdös proved that a finite linear space on v points has at least v lines, with equality occurring if and only if the space is either a near-pencil (all points but one collinear) or a projective plane.

In this paper, we study finite linear spaces which are not near-pencils. We obtain a lower bound for the number of lines (as a function of the number of points) for such linear spaces. A finite linear space which meets this bound can be obtained provided a suitable projective plane exists. We then investigate the converse: can a finite linear space meeting the bound be embedded in a projective plane.

1. Introduction

A *finite linear space* is a pair $(X, \mathscr{B})$, where X is a finite set, and $\mathscr{B}$ is a set of proper subsets of X, such that

(1) every unordered pair of elements of X occurs in a unique $B \in \mathscr{B}$,

(2) every $B \in \mathscr{B}$ has cardinality at least two.

the elements of X are called *points*; members of $\mathscr{B}$ are called *lines* or *blocks*. We will usually let $v = |X|$ and $b = |\mathscr{B}|$. The *length* of a line will be the number of points it contains; the *degree* of a point will be the number of lines on which it lies. We will abbreviate the term 'finite linear space' to FLS.

A linear space in which one line contains all but one of the points (and hence all other lines are of length two) is called a *near-pencil*. An FLS which is not a near-pencil is said to be *non-degenerate*. A non-degenerate FLS will be denoted NLS.

A *projective plane* of *order* n is an FLS having n^2+n+1 points and lines, in

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which every line has length $n+1$. A projective plane of order n is known to exist when n is a prime power.

An *affine plane* of *order n* is an NLS having n^2 points and n^2+n lines, in which every line has length n. Affine and projective planes of order n are co-extensive.

A well-known theorem of De Bruijn and Erdös [1] states that in an FLS the relation $b \geqslant v$ holds, with equality if and only if the space is either a near-pencil or a projective plane.

In this paper we obtain similar results for NLS. In an NLS having $v \geqslant 5$ points, we show that $b \geqslant B(v)$, where

$$(*) \qquad B(v)=\begin{cases} n^2+n+1 & \text{if } n^2+2 \leqslant v \leqslant n^2+n+1, \\ n^2+n & \text{if } n^2-n+3 \leqslant v \leqslant n^2+1, \\ n^2+n-1 & \text{if } v=n^2-n+2. \end{cases}$$

Equality can be attained if n is the order of a projective plane.

An NLS is said to be *minimal* if no NLS on v points has fewer lines. We consider the embeddability of minimal NLS with $b=B(v)$ lines in projective planes, and prove several results. For example, if $v=n^2-\alpha$, for some integer n, with $\alpha \geqslant 0$ and $\alpha^2+\alpha(2n-3)-(2n^2-2n) \leqslant 0$, then a minimal NLS with v points and $B(v)$ lines can be embedded into a projective plane of order n. Minimal NLS with $v=n^2-n+2$ $(v>8)$ and $b=n^2+n-1$, can likewise be embedded.

2. Some preliminary results

We require the notion of an $(r, 1)$-design. An $(r, 1)$-*design* is a pair $(X, \mathscr{B})$ where X is a finite set of points, and $\mathscr{B}$ is a family of proper subsets of X called blocks satisfying:

(1) every point occurs in precisely r blocks,

(2) every pair of points occurs in a unique block.

As before we will use v and b to denote respectively the number of points and blocks. By deleting blocks of length one from an $(r, 1)$-design one obtains an FLS, and conversely, given an FLS, the addition of sufficiently many blocks of length one will produce an $(r, 1)$-design for some r.

An $(r, 1)$-design $(X, \mathscr{B})$ is said to be embedded in an $(r, 1)$-design $(X', \mathscr{B}')$ if

(1) $X \subseteq X'$, and

(2) $\mathscr{B}=\{B \cap X \colon B \in \mathscr{B}'\}$

(note $\mathscr{B}$ and $\mathscr{B}'$ are multisets). We will make use of the following results concerning embeddability of $(r, 1)$-designs.

Lemma 2.1. (1) *Suppose an* $(n+1, 1)$-*design D with* v *points and* $b \leqslant n^2+n+1$ *blocks has a point which occurs in s blocks of length n. Then D can be embedded in an* $(n+1, 1)$-*design* D^* *having* $v+s$ *points and at most* n^2+n+1 *blocks.*

(2) *Any* $(n+1, 1)$-*design with* $v \geq n^2$ *points and* $b \leq n^2+n+1$ *blocks can be embedded in a projective plane of order n.*

Proof. See [4]. □

An FLS is defined to be embedded in a larger FLS analogously.

Lemma 2.2. *An* NLS *with* $v \geq n^2$ *points is embeddable in a projective plane of order n if and only if it has at most* n^2+n+1 *lines.*

Proof. See [5]. □

The following two arithmetic results will be of use.

Lemma 2.3. *Given an* FLS *which has the longest line of length k, the inequalities*

$$(1) \quad b \geq 1 + \frac{k^2(v-k)}{v-1} \quad \text{and} \quad (2)\ b \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \right\rceil \right\rceil$$

must hold, where as usual, $\lceil x \rceil$ *denotes the least integer no less than x.*

Proof. (1) is proved in Stanton and Kalbfleisch [3]. (2) is easily proved since every point has degree at least $\lceil (v-1)/(k-1) \rceil$. □

Lemma 2.4. *Suppose* $k_1, \ldots, k_b$ *are non-negative integers, and* $\sum_{i=1}^{b} k_i \geq qb + r$ *where* $0 \leq r < b$ *and* $q \geq 1$. *Then*

$$\sum_{i=1}^{b} \binom{k_i}{t} \geq r\binom{q+1}{t} + (b-r)\binom{q}{t},$$

with equality if and only if precisely r of the k_i*'s equal* $q+1$ *and the remaining* k_i*'s equal q (hence* $\sum_{i=1}^{b} k_i = qb + r$*).*

Proof. See [2]. □

For $v \geq 4$, denote by $h(v)$ the number of lines in a minimal NLS having v points. We seek to determine the behaviour of the function $h(v)$. This we shall do mainly in the next section, but we first prove a couple of simple results here.

Lemma 2.5. $h(4) = h(5) = 6$.

Proof. Trivial. □

Lemma 2.6. *For* $v \geq 4$, $h(v+1) \geq h(v)$.

Proof. The result is true for $v-4$ by Lemma 2.5. Thus, let F be a minimal NLS on $v+1$ points, $v \geqslant 5$. If F contains no line of length $v-1$, the result is clearly true, so suppose F contains such a line l. For any other line l' of F, the sum of the lengths of l and l' does not exceed $v+2$, so l' has length at most 3. Since $v \geqslant 5$, l is the unique line of length $v-1$. Then we may delete any point x of l from F, and also delete any 'lines' of length one produced by this operation, to obtain an NLS on v points having at most $h(v+1)$ lines. Thus $h(v+1) \geqslant h(v)$, as required. □

3. Minimal non-degenerate finite linear spaces

Let $f(k, v) = 1 + k^2(v-k)/(v-1)$. We have the following.

Lemma 3.1. *If an* FLS *has a longest line of length* k, *and* $2 \leqslant k_1 \leqslant k \leqslant k_2 \leqslant v-2$, *then*

$$b \geqslant \min\{f(k_1, v), f(k_2, v)\}.$$

Proof. Apply Lemma 2.3(1). As observed in [2], the function $f(x, v)$, for fixed v, is unimodal on the interval $[2, v-2]$, having its maximum at $x = \frac{2}{3}v$. □

For future reference, we record some values of the function f.

Lemma 3.2.

(1) $$f(v-2, v) = 2v - 1 + \frac{2}{v-1}.$$

(2) $$f(n+2, n^2+2) = n^2 + n + \frac{2}{n^2+1}.$$

(3) $$f(n+1, n^2+2) = n^2 + 3n - \frac{7n-1}{n^2+1}.$$

(4) $$f(n+2, n^2-n+2) = n^2 + 3n - 1 - \frac{13n-2}{n^2-n+1}.$$

(5) $$f(n+1, n^2-n+2) = n^2 + n - 1 - \frac{3n-3}{n^2-n+1}.$$

Lemma 3.3. *Suppose* $v \geqslant n^2+2$ *and* $n \geqslant 2$. *If an* NLS *on* v *points has a line of length* $n+2$, *then* $b \geqslant n^2+n+2$.

Proof. Clearly $f(v, k)$ is monotone increasing in v for fixed k, and also $f(v-1, v+1) < f(v-2, v)$ for all admissable v. Thus, by Lemma 3.1, we have

$$b \geqslant \min\{f(n+2, n^2+2), f(n^2, n^2+n)\}.$$

If $n \geqslant 2$, then $f(n+2, n^2+2) \leqslant f(n^2, n^2+2)$, so $b \geqslant f(n+2, n^2+2)$. By Lemma 3.2(2), we have

$$f(n+2, n^2+2) = n^2+3n-\frac{7n-1}{n^2+1} = n^2+n+1+\frac{2n^3-n^2-5n}{n^2+1}.$$

For $n \geqslant 2$, $2n^3 > n^2+5n$, so the result follows. □

By a similar argument, one can prove the following

Lemma 3.4. *Suppose $v \geqslant n^2-n+2$ and $n \geqslant 3$. If an* NLS *on v points has a line of length $n+2$, then*

(1) $b \geqslant n^2+n+1$ *if* $n \geqslant 4$,

(2) $b \geqslant n^2+n$ *if* $n=3$.

Proof. As in Lemma 3.3,

$$\begin{aligned} b &\geqslant f(n+2, n^2-n+2) \\ &= n^2+3n-1-\frac{13n-2}{n^2-n+1} \\ &= n^2+n+\frac{2n^3-3n^2-10n+1}{n^2-n+1}. \end{aligned}$$

For $n \geqslant 4$, $2n^3 > 3n^2+10n-1$, which establishes (1). To prove (2), we note that $f(5, 11) > 11$, so $b \geqslant 12$. □

Lemma 3.5. *Suppose $v \geqslant n^2+1$ and $n \geqslant 2$. If an* NLS *on v points has no line of length exceeding n, then $b \geqslant n^2+2n+2$.*

Proof. From Lemma 2.3(2), we obtain

$$b \geqslant \left\lceil \frac{n^2+1}{n} \left\lceil \frac{n^2}{n-1} \right\rceil \right\rceil = \left\lceil \frac{(n^2+1)}{n}(n+2) \right\rceil = n^2+2n+2. \qquad \square$$

Theorem 3.6. *If an* NLS *has $n^2+2 \leqslant v \leqslant n^2+n+1$ for some $n \geqslant 2$, then $b \geqslant n^2+n+1$, with equality holding if and only if the* NLS *can be embedded in to a projective plane.*

Proof. Let F be such an NLS. If the longest line in F has length other than $n+1$, then $b \geqslant n^2+n+2$ by Lemmata 3.3 and 3.5. Also,

$$f(n+1, n^2+2) = n^2+n+\frac{2}{n^2+1},$$

so $b \geqslant n^2+n+1$. If, however, F has $b = n^2+n+1$, then F can be embedded in a projective plane by Lemma 2.2. Conversely, if one deletes $n^2+n+1-v$ points from a projective plane of order n, then an FLS with $b=n^2+n+1$ is obtained. □

Lemma 3.7. *If an* NLS *F has $v=n^2-n+2$ for some $n\geq 3$, then $b\geq n^2+n-1$ with equality only if F contains a unique longest line of length $n+1$.*

Proof. First, assume F has at most n^2+n-1 lines, each of which has length not exceeding n. Let $x_1, \ldots, x_v$ denote the points, and let $l_1, \ldots, l_b$ denote the lines of F. For $1\leq i\leq v$, let r_i denote the degree of x_i, and for $1\leq i\leq b$, let k_i denote the length of l_i. Also, let $b^*=n^2+n-1$, and, if $b<b^*$, let $k_i=0$ for $b+1\leq i\leq b^*$.

We have, for $1\leq i\leq v$,

$$r_i\geq\left\lceil\frac{n^2-n+1}{n-1}\right\rceil=n+1.$$

then

$$\sum_{i=1}^{b^*} k_i=\sum_{i=1}^{p} r_i\geq(n^2-n+2)(n+1).$$

We have $(n^2-n+2)(n+1)=(n-1)(n^2+n-1)+3n+1$, and $\sum_{i=1}^{b^*}\binom{k_i}{2}=\binom{v}{2}$. Thus Lemma 2.4 implies

$$(n^2-n+2)(n^2-n+1)\geq(3n-1)(n)(n-1)+(n^2-2n-2)(n-1)(n-2),$$

or

$$n^4-2n^3+4n^2-3n+2\geq n^4-2n^3+4n^2+n-4$$

or $4n\leq 6$, a contradiction.

Hence if F has no line of length $n+1$, then by Lemma 3.4 and the above, F has at least n^2+n lines. So assume F has a line l of length $n+1$. We have

$$f(n+1, n^2-n+2)=n^2+n-1-\frac{3n-3}{n^2-n+1},$$

so for $n\geq 3$, F has at least n^2+n-1 lines. We wish to show that if F has exactly n^2+n-1 lines, then l is the only line of length $n+1$.

Suppose l^* is another line of length $n+1$. If l and l^* contain no common point, then $b\geq(n+1)^2+2>n^2+n-1$, a contradiction, so we may assume $l\cap l^*=\{x_1\}$. Then, for $i>1$, $r_i\geq n+1$. Also, $r_1\geq\lceil(n^2-n+1)/n\rceil=n$. Counting lines which intersect l, we obtain $b\geq n+n\cdot n=n^2+n$, a contradiction. Thus l is the unique line of length $n+1$ in F. □

Lemma 3.8. *Let F be an* NLS *with $v=n^2-n+2$ and $b=n^2+n-1$ for some $n\geq 4$. Then F can be embedded in a projective plane of order n.*

Proof. By the previous lemma, F contains a unique line $l=l_b$ of length $n+1$.

Also, if $x_i \in l$ then $r_i \geqslant n$, and if $x_i \notin l$, then $r_i \geqslant n+1$. Consider

$$\begin{aligned}&(n^2-n+2)(n^2-n+1)\\&\quad=(n+1)n+(3n-3)\cdot n(n-1)+(n-1)^2(n-1)(n-2).\end{aligned}$$

Thus F has at least $3n-3$ lines of length n, with equality occurring if and only if the remaining lines (excluding l) have length $n-1$. For $1 \leqslant i \leqslant b-1$, let

$$k_i' = \begin{cases} k_i & \text{if } |l_i \cap l| = 0, \\ k_i - 1 & \text{if } |l_i \cap l| = 1. \end{cases}$$

Then

$$\sum_{i=1}^{b-1} k_i' \geqslant (n^2-2n+1)(n+1).$$

However

$$(n^2-2n+1)(n+1)=(n-2)(n^2+n-2)+3n-3.$$

Thus, by Lemma 2.4,

$$\begin{aligned}(n^2-2n+1)(n^2-2n) &\geqslant (3n-3)(n-1)(n-2)+(n^2-2n+1)(n-2)(n-3)\\&=(n^2-2n+1)(n^2-2n).\end{aligned}$$

Therefore F contains at most $3n-3$ lines of length n. By the remarks above, F contains one line of length $n+1$, $3n-3$ lines of length n, and n^2-2n+1 lines of length $n-1$. Also, the line of length $n+1$ meets every other line.

Now let x be any point on l, and let a_i denote the number of lines of length i through x, for $n-1 \leqslant i \leqslant n+1$. Then

$$(n-2)a_{n-1}+(n-1)a_n=n^2-2n+1$$

and $a_{n+1}=1$, so either $(a_{n-1}, a_n, a_{n+1})=(0, n-1, 1)$ or $(n-1, 1, 1)$, since n is at least 4. Thus x lies on either n or $n+1$ lines.

Since l meets every other line, we have

$$1+\sum_{x_i \in l}(r_i-1)=n^2+n-1.$$

Thus there are precisely two points x_1 and x_2 of l which have degree n. By adjoining blocks $\{x_1\}$ and $\{x_2\}$ we obtain an $(n+1, 1)$ design with n^2-n+2 points and n^2+n+1 blocks. Also, x_1 lies on $n-1$ lines of length n. Applying Lemma 2.1, we can embed F in an $(n+1, 1)$ design on n^2+1 points and n^2+n+1 blocks, which can in turn be embedded in a projective plane of order n. Hence F can be embedded in a projective plane of order n. □

Lemma 3.9. *Let F be an* NLS *having eight points and eleven lines. Then either F can be embedded in the projective plane of order* 3, *or F is isomorphic to the linear space in Fig.* 1 *below.*

Proof. If all points of F have degree at most 4, then as in the previous lemma, F

can be embedded in a projective plane of order 3. However, for $n=3$ (in Lemma 3.8) there is an additional possibility for the vector (a_2, a_3, a_4), namely $(4, 0, 1)$. Should F contain a point ∞ having this distribution, all other points have degree 3. We may easily construct F, and verify that it is unique up to isomorphism. The unique such F is exhibited in Fig. 1 below.

∞123	145	246	347
∞4	167	257	356
∞5			
∞6			
∞7			

Fig. 1.

Theorem 3.10. *For $n \geqslant 3$, there exists an* NLS *with $v = n^2 - n + 2$ and $b = n^2 + n - 1$ if and only if n is the order of a projective plane.*

Proof. In view of Lemmata 3.8 and 3.9, is suffices to show that if n is the order of a projective plane, then the desired NLS exists. Let π be any projective plane of order n; and l_1 and l_2 be two lines of π. For $i = 1, 2$, let x_i be a point of l_i other than $l_1 \cap l_2$. Then delete from π the points of $l_1 \cup l_2 \setminus \{x_1, x_2\}$, and also delete the lines l_1 and l_2. The resulting NLS has $n^2 - n + 2$ points and $n^2 + n - 1$ lines. □

Lemma 3.11. *Let F be an* NLS *with $v \geqslant n^2 - n + 3$ for some $n \geqslant 3$. Then $b \geqslant n^2 + n$, with equality only if the longest line in F has length n or $n+1$.*

Proof. First suppose that F has a line of length at least $n+2$. If $n \geqslant 4$, then Lemma 3.4 implies the result. If $n = 3$, then we compute $f(5, 9) = 27/2$, so $b \geqslant 14$, and the result is true here as well.

Next, suppose F has no line of length exceeding $n-1$. Then by Lemma 2.3(2),

$$b \geqslant \left\lceil \frac{n^2-n+3}{n-1} \left\lceil \frac{n^2-n+2}{n-2} \right\rceil \right\rceil \geqslant \left\lceil \frac{(n^2-n+3)(n+2)}{n-1} \right\rceil > n^2+n+1.$$

Next, suppose F has a longest line of length n. Every point has degree at least $\lceil (n^2-n+2)/(n-1) \rceil = n+1$. An application of Lemma 2.4 yields $b > n^2+n-1$ when $v = n^2 - n + 3$.

Finally, we consider the case where the longest line l has length $n+1$. If l is the only line of length $n+1$, then every point on l has degree at least $1 + \lceil (n^2-2n+2)/(n-1) \rceil = n+1$, and $b \geqslant 1 + (n+1)n = n^2 + n + 1$. So assume l^* is another line of length $n+1$. If l and l^* are disjoint then $b \geqslant (n+1)^2 + 2$, so assume l and l^* meet in a point x. The point x has degree at least $\lceil (n^2 - n + 2)/n \rceil = n$, and any other point of F has degree at least $n+1$. Thus $b \geqslant 1 + n - 1 + n \cdot n = n^2 + n$, and the result follows by the monotonicity of the function h.

Corollary 3.12. *If F is an* NLS *with $v \geq n^2-n+3$ and $b=n^2+n$, for some $n \geq 3$, and if the longest line in F has length $n+1$, then one point has degree n and all other points have degree $n+1$.*

Proof. In order to attain $b=n^2+n$ in the above lemma, we must have

(1) all lines of length $n+1$ meet at a point x of degree n, and

(2) any line meets all lines of length $n+1$.

Thus x has degree n and all other points have degree $n+1$. □

Such a situation can be realized if n is the order of a projective plane.

Lemma 3.13. *Suppose $n \geq 3$ is the order of a projective plane and $n^2-n+3 \leq v \leq n^2$. Then there exists an* NLS *having v points and $b=n^2+n$ lines, in which the longest line has length n or $n+1$, as desired.*

Proof. Let π be a projective plane of order $n \geq 3$, and let $v=n^2+n+1-\alpha$, where $n+1 \leq \alpha \leq 2n-2$.

Let l_1 and l_2 be two lines of π, which meet in a point x. If we delete all points of l_1, and $\alpha-(n+1)$ points from $l_2 \setminus \{x\}$ we obtain an NLS with n^2+n lines, in which the longest line has length n. If we delete the points of $l_1 \setminus \{x\}$ and $\alpha-n$ points of $l_2 \setminus \{x\}$, we obtain an NLS with n^2+n lines, in which the longest line has length $n+1$. □

When $v=n^2+1$, we have the following.

Lemma 3.14. *If an* NLS *on n^2+1 points has n^2+n lines, then the longest line has length $n+1$, and the space can be embedded into a projective plane of order n. Conversely, if n is the order of a projective plane, then $h(n^2+1)=n^2+n$.*

Proof. We have $h(n^2+1) \geq n^2+n$. Suppose π is a projective plane of order n. Let l be any line, and let x be any point of l. If we delete all points of $l \setminus \{x\}$, and the line l, from π, we obtain an NLS with $v=n^2+1$ and $b=n^2+n$, having a line of length $n+1$.

Now suppose F is an NLS with $b=n^2+1$ and $b=n^2+n$. We have established (Lemma 3.11) that the longest line of F has length n or $n+1$. The first case is ruled out by Lemma 3.5, so the longest line has length $n+1$. Finally, F can be embedded in a projective plane by Lemma 2.2. □

We now consider the embeddability of NLS on v points and n^2+n lines where $n^2-n+2 \leq v \leq n^2$, in projective planes. We first consider the case where the longest line is of length n.

Let G be an FLS. A set $\mathscr{L}$ of lines is said to *span* F if for any line l in F there exists a line $l_1 \in \mathscr{L}$ such that l and l_1 contain a point in common. Now, suppose T

is a set of lines such that any two distinct intersecting lines in T span F. Let U be the set of lines of F that are disjoint from at least one line of T. For each l in T, let $D(l)$ denote the set of all lines of U disjoint from l, and let $E(l)=D(l)\cup\{l\}$. Define a relation $\sim$ on $S=T\cup U$ by the rule $a\sim b$ if there exists $l\in T$ such that $\{a,b\}\subseteq E(l)$.

Lemma 3.15. *If $E(l_1)=E(l_2)$ whenever $l_1\cap l_2=\emptyset$, then $\sim$, as described above, is an equivalence relation on S.*

Proof. Suppose l_1 and l_2 intersect, for distinct $l_1, l_2\in T$. Since $\{l_1, l_2\}$ spans F, therefore $E(l_1)\cap E(l_2)=\emptyset$.

Now, suppose $a\sim b$ and $b\sim c$. Let $\{a,b\}\subseteq E(l_1)$ and $\{b,c\}\subseteq E(l_2)$ for some l_1, l_2. If l_1 and l_2 are disjoint or equal, then $E(l_1)=E(l_2)$ so $\{a,c\}\subseteq E(l_1)$ and $a\sim c$. If l_1 and l_2 are distinct and intersect, then $E(l_1)\cap E(l_2)=\emptyset$, so we cannot have $b\in E(l_1)\cap E(l_2)$. □

Lemma 3.16. *Let F be an* NLS *with $v\geqslant n^2-n+2$ and $b=n^2+n$ in which the longest line has length n. Let T denote the set of lines of length n. Then $\sim$ is an equivalence relation on the set S as described above.*

Proof. We must show that

(1) any pair of distinct intersecting lines l_1 and l_2 of length n span F, and

(2) if l_1 and l_2 are disjoint lines of length n and any line l is disjoint from l_1, then l is disjoint from l_2.

First, we note that every point in F has degree at least $\lceil (n^2-n+1)/(n-1)\rceil = n+1$.

Let x be any point on a line l of the length n. If x has degree greater than $n+1$, then there are at most $n^2+n-(1+n\cdot n+1)=n-2$ lines disjoint from l. Thus the lines disjoint from l have average length at least $(n^2-2n+3)/(n-2)>n$, so some line has length greater than n, a contradiction. Therefore every point on a line of length n has degree $n+1$.

Let l_1 and l_2 be distinct intersecting lines of length n. Since every point on l_1 and l_2 has degree $n+1$, the number of lines spanned by l_1 and l_2 is at least $n+1+(n-1)^2+2(n-1)=n^2+n$. Since $b=n^2+n$, l_1 and l_2 span all lines. This proves (1).

Now, let l_1 and l_2 be disjoint lines of length n. Suppose a line l intersects l_2 in a point x. The point x has degree $n+1$, and l_2 has length n, so there is a unique line through x which is disjoint from l_2, namely, l_1. Thus l intersects l_1, which proves (2). □

Let F be an NLS satisfying the hypotheses of Lemma 3.16, which has $v=n^2-\alpha$ points $(0\leqslant\alpha\leqslant n-2)$. Let $P_1,\ldots,P_s$ denote the equivalence classes (with respect to the relation $\sim$), and let W denote the lines of F which are in no P_i, $1\leqslant i\leqslant s$.

Now every point has degree at least $n+1$. Denote the degree of x by $n+\beta_x$ where $\beta_x \geqslant 1$ for all points x. Let $\delta = \sum_x \beta_x - v$.

Lemma 3.17. *The number of equivalence classes s satisfies*

$$s \geqslant 1 + \frac{n(n-\alpha)}{n-\alpha+\delta}.$$

Proof. Let x be any point. Then in any P_i, there are β_x lines containing x. Thus

$$\sum_{l \in P_i} k_l = \sum_x \beta_x = v + \delta, \quad \text{for any } i,$$

where k_l denotes the length of the line l. Then

$$\sum_{l \in W} k_l = (n+1)v + \delta - s(\delta + v).$$

Next we note that every P_i contains precisely n lines. This follows since a line of length n spans n^2+1 lines, and is therefore disjoint from $n-1$ lines, since each point on a line of length n has degree $n+1$. Thus $|W| = n^2+n-sn$.

Now, each line in W has length at most $n-1$, since the lines of W occur in no P_i. Thus

$$\sum_{l \in W} k_l \leqslant (n-1)\,|W|.$$

Substituting, we obtain

$$(n+1)v + \delta - s(\delta+v) \leqslant (n-1)(n^2 - n(s-1)).$$

Thus

$$(n+1)v + \delta - (n-1)(n^2+n) \leqslant s(v+\delta-n^2+n).$$

Since $v = n^2 - \alpha$, we obtain

$$n^2 - \alpha n + n - \alpha + \delta \leqslant s(n - \sigma + \delta),$$

so

$$s \geqslant 1 + \frac{n(n-\alpha)}{n-\alpha+\delta}. \qquad \square$$

Lemma 3.18. *An* $(n+1, 1)$*-design F on* $v = n^2 - \alpha$ *points* $(0 \leqslant \alpha \leqslant n-2)$, *which has* n^2+n *lines, can be embedded into a projective plane of order n.*

Proof. Consider the classes $P_1, \ldots, P_s$. Since $\delta = 0$, therefore, by the proof of Lemma 3.17, $s = n+1$ and $W = \emptyset$. Each P_i consists of n lines which partition the point set. Let $\infty_1, \ldots, \infty_{n+1}$ be $n+1$ new points. For $1 \leqslant i \leqslant n+1$, adjoin ∞_i to each line of P_i, and adjoin the line $\infty_1\infty_2 \cdots \infty_{n+1}$. The NLS thus constructed has n^2+n+1 lines and at least n^2 points, and so can be embedded into a projective plane of order n. This establishes the lemma. $\square$

Theorem 3.19. *Suppose F is an* NLS *with $v=n^2-\alpha$ points $(0\leqslant\alpha\leqslant n-3)$ and n^2+n lines, the longest of which has length $n+1$. Then F can be embedded into a projective plane of order n.*

Proof. In the proof of Corollary 3.12, we have noted that all lines of length $n+1$ pass through a point (say ∞), and that all other points have degree $n+1$. The linear space F' obtained by deleting ∞ from F is an $(n+1)$-design which satisfies 4
the hypotheses of Lemma 3.18. Hence F' can be embedded into a projective plane π of order n. It is also clear that the lines of F' which passed through ∞ (in F) form one of the classes P_i, so that the point ∞ is restored during the embedding of F' into π. Hence F can be embedded into π. $\square$

We now return to the case of linear spaces with $n^2-\alpha$ points and n^2+n lines, the longest of which has length n. As before, we let point x have degree $n+\beta_x$ and denote $\delta=\sum\beta_n-v$.

Lemma 3.20. *If $\delta>0$, then*

$$\delta\geqslant\begin{cases} n-\alpha & \text{if } n \text{ odd},\\ (n-\alpha)\left(\dfrac{n+1}{n-1}\right) & \text{if } n \text{ even}.\end{cases}$$

Proof. Recall that s denotes the number of equivalence classes P_i, and $s\geqslant 1+n(n-\alpha)/(n-\alpha+\delta)$ by Lemma 3.17. Since there is a point x with $\beta_x\geqslant 2$, and since x occurs β_x times in each P_i, then counting lines through x yields $s\beta_x\leqslant n+\beta_x$, or $s\leqslant 1+\lfloor n/\beta_x\rfloor$ where, as usual $\lfloor y\rfloor$ denoted the greatest integer not exceeding y. Since $\beta_x\geqslant 2$, we have $s\leqslant 1+\lfloor\frac{1}{2}n\rfloor$.

Now, if n is even, $\lfloor\frac{1}{2}n\rfloor=\frac{1}{2}n$, and we have

$$1-\frac{n(n-\alpha)}{n-\alpha+\delta}\leqslant 1+\tfrac{1}{2}n,$$

so $2(n-\alpha)\leqslant n-\alpha+\delta$ and $\delta\geqslant n-\alpha$. If n is odd, then $\lfloor\frac{1}{2}n\rfloor=\frac{1}{2}(n-1)$ and we obtain $\delta\geqslant(n-\alpha)(n+1)/(n-1)$ similarly. $\square$

We now obtain an upper bound for δ.

Lemma 3.21. $\delta\leqslant(\alpha^2-\alpha)/2(n-1)$.

Proof. We have

$$\sum_l k_l=(n+1)v+\delta=(n-1)(n^2+n)+r,$$

where $r=(n-\alpha)(n+1)+\delta$. Note that $r\leqslant n^2+n$, for otherwise the average line length would be at least n, which is an impossibility. We apply Lemma 2.4 with $q=n-1$, $b=n^2+n$, and $t=2$.

Since $\sum_l \binom{k_l}{2} = \binom{v}{2}$, we obtain

$$v(v-1) \geq rn(n-1) + (b-r)(n-1)(n-2).$$

If we substitute $v=n^2-\alpha$, $b=n^2+n$, and $r=(n-\alpha)(n+1)+\delta$ and simplify, the desired result is obtained. □

We now combine the bounds of the two previous lemmata.

Lemma 3.22. *Suppose $\delta > 0$. If n is even, then*

$$\alpha^2+\alpha(2n-3)-(2n^2-2n) \geq 0.$$

If n is odd, then

$$\alpha^2+\alpha(2n+1)-(2n^2+2n) \geq 0.$$

Theorem 3.23. *Suppose F is an* NLS *with $n^2-\alpha$ points ($\alpha \geq 0$) and n^2+n lines, the longest of which has length n. If n is even and $\alpha^2+\alpha(2n-3)-(2n^2-2n)<0$, or if n is odd and $\alpha^2+\alpha(2n+1)-(2n^2+2n)<0$, then F can be embedded in a projective plane of order n.*

Proof. From Lemma 3.22, $\delta=0$, so F is an $(n+1, 1)$-design and can be embedded in a projective plane of order n by Lemma 3.18. □

Corollary 3.24. *If F is an* NLS *on v points and $B(v)$ lines, where $9 \leq v \leq 134$, then F can be embedded in a projective plane of order n (where $n^2-n+2 \leq v \leq n^2+n+1$).*

Proof. The proof follows from Theorem 3.6, Lemma 3.8, Lemma 3.14, Theorem 3.19 and Theorem 3.23. The first instance when the hypotheses of Theorem 3.23 are violated is $n=12$ and $\alpha=9$. □

5. Open problems

There are several open questions which arise in connection with finite linear spaces. Doyen has asked, given v, the number of points, what are the possible values for b, the number of lines? In this regard, P. Erdös and V.T. Sós have shown that there is an absolute constant c so that for every b satisfying

$$cv^{3/2} < b \leq \binom{n}{2}, \qquad b \neq \binom{v}{2} - i, \quad i=1,3,$$

will occur as the number of lines. (This result is best possible part from the value of c.)

Let $(k_1, k_2, \ldots, k_b)$ be a set of integers such that each $k \geq 2$ and $\sum k_i(k_i - 1) = v(v-1)$ for some integer v. Give reasonable necessary and sufficient conditions that there exists a finite linear space on points whose line lengths are specified by the k_i.

Let $(r_1, r_2, \ldots, r_v)$ be a set of positive integers such that each $r_i \geq 2$. Give reasonable necessary and sufficient conditions that there exist a finite linear space on v points such that the ith point lies on precisely r_i lines. (These questions are clearly very difficult and probably cannot be answered with 'side' conditions.)

Given a finite linear space F with v points and b lines satisfying $v \leq b \leq n^2+n+1$ for some positive integer n, then for v large, all points of F must lie on no more than $n+1$ points. Given n, is the largest value of v such that there exists a finite linear space on v points which contains a point which lies on at least $n+2$ lines? We conjecture that such a v must be less than n^2-n+2 for $n>3$.

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