nternat. J. Math. & Math. Sci.
Vol. 6 No. 3 (1983) 409-418

DOT PRODUCT REARRANGEMENTS

PAUL ERDOS and GARY WEISS

Mathematics Institute, Budapest; University of Cincinnati

(Received September 18, 1980)

ABSTRACT. Let $a=(a_n)$, $x=(x_n)$ denote nonnegative sequences; $x=(x_{\pi(n)})$ denotes the rearranged sequence determined by the permutation π, $a\cdot x$ denotes the dot product $\Sigma a_n x_n$; and $S(a,x)$ denotes $\{a\cdot x_\pi : \pi$ is a permuation of the positive integers$\}$. We examine $S(a,x)$ as a subset of the nonnegative real line in certain special circumstances. The main result is that if $a_n \uparrow \infty$, then $S(a,x) = [a\cdot x,\infty]$ for every $x_n \downarrow\neq 0$ if and only if a_{n+1}/a_n is uniformly bounded.

KEY WORDS AND PHRASES. *Dot product, series rearrangements, conditional convergence.*

1982 MATHEMATICS SUBJECT CLASSIFICATION CODE. 40A05.

An elementary classical result of Riemann on infinite series states that a conditionally convergent series that is not absolutely convergent can be rearranged to sum to any extended real number. A slightly similar group of questions arose in connection with certain formulas in operator theory [1, p. 181]. Namely, if we let $a=(a_n)$, $x=(x_n)$ denote any two non-negative sequences and x_π denote the sequence $(x_{\pi(n)})$ where π is any permutation of the positive integers, then what can be said about the set of non-negative real numbers $S(a,x) = \{a\cdot x : \pi$ is a permutation of the positive integers$\}$. More specifically, which subsets of the non-negative real line can be realized as the form $S(a,x)$ for some such a and x?

Various facts about $S(a,x)$ are obvious

(1) $S(a,x) \subset [0,\infty]$. The values 0 and ∞ may be obtained.

(2) If a and x are strictly positive sequences or are at most finitely zero, then $S(a,x) \subset (0,\infty]$.

(3) Not all subsets of $[0,\infty]$ are realizable as an $S(a,x)$ set. This follows by a cardinality argument. If c denotes the cardinality of $[0,\infty]$, then the cardinality of the class of subsets of $[0,\infty]$ is 2^c, but the cardinality of the class of sequences a and x is c and thus the cardinality of the subsets $S(a,x)$ is less than or equal to $c \cdot c = c$.

(4) If either a or x is finitely non-zero then $S(a,x)$ is countable.

(5) An example: if $a = (0,2,0,2,\ldots)$ and $x = (3^{-n})$, then $S(a,x)$ is precisely the Cantor set except for those non-negative real numbers whose ternary expansion consists of a tail of 0's or a tail of 2's (i.e., a subset of the rational numbers.),

It seems too ambitious to consider the general question at this time. For this reason we shall restrict our attention to the cases when a is a non-decreasing sequence and x is a non-increasing sequence,

If $a \equiv 0$ or $x \equiv 0$, the problem is trivial and $S(a,x) = \{0\}$. If $a_1 \neq 0$ and $x_n \nrightarrow 0$, the problem is trivial and $S(a,x) = \{\infty\}$. If a_n is bounded by M, then $S(a,x) \subset [0, M\sum x_n]$. In any case, hereafter we shall assume $a_n \uparrow \infty$ and $x_n \downarrow 0$, unless otherwise specified.

The Lemma that follows is a well-known fact, but we give a proof for completeness and because the proof contains some of the ideas used in the main result.

LEMMA. If $a_n \uparrow$ and $x_n \downarrow$ then $S(a,x) \subset [a \cdot x, \infty]$. In addition, $a \cdot x \in S(a,x)$, and if $a_n \uparrow \infty$ and $x_n \neq 0$ for all n or if $a_n \uparrow$ and $a_n > 0$ for some n and $x_n \neq 0$, then $\infty \in S(a,x)$.

PROOF. It suffices to show that for every permutation π of the positive integers, we have $a \cdot x \leq \sum a_n x_{\pi(n)}$ or, equivalently, $a \cdot x \leq \sum a_{\pi(n)} x_n$ for every π. The rest of the lemma is clear.

Define π_1 in terms of π as follows. Set

$$\pi_1(n) = \begin{cases} 1 & n = 1 \\ \pi(1) & n = \pi^{-1}(1) \\ \pi(n) & \text{otherwise} \end{cases}$$

It is straightforward to verify that π_1 is also a permutation of the positive integers (one-to-one and onto) which fixes 1. We assert that $a_{\pi_1} \cdot x \leq a_\pi \cdot x$ to see this, note that $\pi(1) \geq 1$ and $\pi^{-1}(1) \geq 1$. Hence $a_{\pi(1)} - a_1 \geq 0$ and $x_1 - x_{\pi^{-1}(1)} \geq 0$. Therefore

$$\begin{aligned} \sum(a_{\pi(n)} - a_{\pi_1(n)})x_n &= (a_{\pi(1)} - a_{\pi_1(1)})x_1 + (a_{\pi(\pi^{-1}(1))} - a_{\pi_1(\pi^{-1}(1))})x_{\pi^{-1}(1)} \\ &= (a_{\pi(1)} - a_1)(x_1 - x_{\pi^{-1}(1)}) \\ &\geq 0 . \end{aligned}$$

Proceeding inductively, we obtain a sequence of permutations π_k that fix $1,2,\ldots,k$ for which $a_{\pi_k} \cdot x \leq a_{\pi_{k-1}} \cdot x$. Hence, for every k ,

$$\sum_{n=1}^{k} a_n x_n = \sum_{n=1}^{k} a_{\pi_k(n)} x_n \leq a_{\pi_k} \cdot x \leq a_\pi \cdot x .$$

Letting $k \to \infty$, we obtain $a \cdot x \leq a_\pi \cdot x$.

The main question of this paper is: for which a,x with $a_n \uparrow \infty$ and $x_n \downarrow 0$ is $S(a,x) = [a \cdot x,\infty]$?

The main result of this paper gives a partial answer. Namely, we can characterize which $a_n \uparrow \infty$ have the property that $S(a,x) = [a \cdot x,\infty]$ for every x such that $x_n \underset{\neq}{\downarrow} 0$.

On first sight, it might appear that $S(a,x)$ can never be $[a \cdot x,\infty]$ or that it is quite rare. The first result in this direction was that if $a_n = n$ for every n, then $S(a,x) = [a \cdot x,\infty]$ for every x such that $x_n \neq 0$. That $S(a,x)$ may not be $[a \cdot x,\infty]$ was first decided by an example due to Robert Young. Namely, let $a_n = 2^{2^n}$ and $x_n = 2^{-2^{n+1}}$. Both results are unpublished. The succeeding results and techniques are due to the work of the authors in collaboration with Hugh Montgomery.

THEOREM 1. (The Main Theorem) Let $a = (a_n)$ where $a_n > 0$ for every n and $a_n \to \infty$. Consider the following conditions:

(1) a_{n+1}/a_n is bounded.

(2) For the non-negative sequence $x = (x_n)$, there exist subsequences (a_{n_k}) and (x_{m_k}) of a and x respectively such that

(a) $a_{n_k} x_{m_k} \to 0$ as $k \to \infty$, and

(b) $\sum_k a_{n_k} x_{m_k} = \infty$.

(3) $S(a,x) = [a \cdot x, \infty]$.

Then (1) implies (2) for every strictly positive sequence $x = (x_n)$ that tends to 0. Also if $a_n \uparrow \infty$ and $x_n \downarrow 0$ where $a_n, x_n \neq 0$ for all n, then (2) implies (3).

PROOF. To prove that (1) implies that (2) holds for every strictly positive sequence $x = (x_n)$ that tends to 0, suppose $a_{n+1}/a_n \leq M$ for all n. We assert that for every positive integer k, there exist arbitrarily large positive integers n_k and m_k for which $(k+1)^{-1} \leq a_{n_k} x_{m_k} \leq Mk^{-1}$. If this assertion were true, then clearly we could choose two strictly increasing subsequences of positive integers (n_k) and (m_k) such that $a_{n_k} x_{m_k} \to 0$ as $k \to \infty$ to prove the assertion.

For each fixed positive integer k, $(k+1)^{-1} \leq a_n x_m \leq Mk^{-1}$ if and only if $x_m \in [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$. All we need show is that there exist arbitrarily large n,m for which $x_m \in [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$.

Suppose to the contrary that there exists a positive integer N for which $x_m \notin [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$ for every $n,m \geq N$. In other words, for every $m \geq N$ $x_m \notin \bigcup_{n \geq N} [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$. (Note: This would imply that $\bigcup_{n \geq N} [(a_n(k+1))^{-1}, M(a_n k)-1]$ cannot contain any interval of the form $(0,\varepsilon)$ for some $\varepsilon > 0$, since $x_m \to 0$ as $m \to \infty$. However, this is not the case. Indeed, the proof below can be used to show that for every N, there exists $\varepsilon > 0$ such that $(0,\varepsilon) \subset \bigcup_{n \geq N} [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$.).

For each $m \geq N$, let n_m denote the least positive integer n such that $M(a_{n+1}k)^{-1} < x_m$, which exists since $a_n \to \infty$ as $n \to \infty$ and hence $M(a_{n+1}k)^{-1} \to$

as $n \to \infty$. For m sufficiently large, we have $M(a_{n_m+1}k)^{-1} \le x_m \le M(a_{n_k}k)^{-1}$. Also, since $M(a_{n_m+1}k)^{-1} < x_m$ and $x_m \to 0$ as $m \to \infty$, we have $m \to \infty$ implies $a_{n_m+1} \to \infty$ and hence $n_m \to \infty$. Therefore $n_m \ge N$ for all m sufficiently large, and for these m, $x_m \notin [(a_{n_m}(k+1))^{-1}, M(a_{n_m}k)^{-1}]$. Hence, for infinitely many m, we have $x_m \le M(a_{n_k}k)^{-1}$ and $x_m \notin [(a_{n_m}(k+1))^{-1}, M(a_{n_m}k)^{-1}]$. Therefore, for infinitely many m, we have $M(a_{n_m+1}k)^{-1} < x_m < (a_{n_m}(k+1))^{-1}$. This implies that $M(a_{n_m+1}k)^{-1} < (a_{n_m}(k+1))^{-1}$ for infinitely many m, or equivalently, $a_{n_m+1}/a_{n_m} > M(k+1)/k > M$ for infinitely many m, which contradicts our assumption that $a_{n+1}/a_n \le M$ for all n. Hence (2) is proved.

To prove (2) $\to$ (3) whenever $a_n \uparrow \infty$ and $x_n \overset{\downarrow}{\neq} 0$, suppose (2) holds for a and x, so that there exist subsequences (a_{n_k}) and (x_{m_k}) such $a_{n_k}x_{m_k} \to 0$ as $k \to \infty$, and $\sum_k a_{n_k}x_{m_k} = \infty$. We first assert that without loss of generality we may assume that $a \cdot x = \sum a_n x_n < \infty$. To see this suppose $a \cdot x = \sum a_n x_n = \infty$. Then by the lemma we have that $S(a,x) = \{\infty\}$, and hence (3) holds.

Assuming that $\sum a_n x_n < \infty$, we next assert that without loss of generality we can assume that $n_k > m_k$ for every k. To see this, let Z_1 denote the set $\{k : n_k > m_k\}$ and let Z_2 denote the set $\{k : n_k \le m_k\}$. Then

$$\infty = \sum_k a_{n_k}x_{m_k} = \sum_{k \in Z_1} a_{n_k}x_{m_k} + \sum_{k \in Z_2} a_{n_k}x_{m_k}$$

But $\sum_{k \in Z_2} a_{n_k}x_{m_k} \le \sum_{k \in Z_2} a_{n_k}x_{n_k} \le \sum a_n x_n < \infty$. Therefore $\sum_{k \in Z_1} a_{n_k}x_{m_k} = \infty$. Let Z_1 determine subsequences of (n_k) and (m_k), which for simplicity we again call (n_k) and (m_k), respectively, by taking only those entries n_k, m_k (in increasing order) for which $k \in Z_1$. This gives us subsequences (a_{n_k}) and (x_{m_k}) of a and x which satisfy conditions a and b in the 2^{nd} condition of the theorem, and in addition satisfy $n_k > m_k$ for all k.

Next we assert that without loss of generality we may assume $n_k \ne m_j$ for all k,j. To see this, note that we have $n_k > m_k$ for all k and that $\langle n_k \rangle$ and $\langle m_k \rangle$ are strictly increasing (a property of subsequences). Therefore if $n_k = m_j$ for some k,j, then $k < j$ and $n_k \neq m_i$ for all $i \neq j$. That is, n_k can occur at most once among the m_j's. Put $(n_1,m_1),\ldots,(n_{k_1},m_{k_1}) \in S_1$ where k_1+1 is the least positive integer such that $m_{k_1+1} = n_k$ for some $k < k_1 + 1$. Put $(n_{k_1+1}, m_{k_1+1}),\ldots,(n_{k_2},m_{k_2}) \in S_2$ where k_2+1 is the least positive integer, if it exists, such that $m_{k_2+1} = n_k$ for some $k_1+1 \le k < k_2+1$. Put $(n_{k_2+1}, m_{k_2+1}),\ldots,$ $(n_{k_3}, m_{k_3}) \in S_1$ such that k_3+1 is the least positive integer, if it exists, such that $m_{k_3+1} = n_k$ for some $k \le k_1$ or $k_2 \le k < k_3+1$. Continuing in this way, if no such least positive integer exists, then either S_1 or S_2 is finite. Otherwis both S_1,S_2 are infinite. For either case, no $n_k = m_j$ when both $(n_k,m_k),(n_j,m_j)$ $\in S_1$ or S_2. Then clearly S_1,S_2 is a disjoint partition of the set of all $(n_k,m_k$ and in each set, no n_k appears as an m_j. Therefore $\infty = \sum a_{n_k} x_{m_k} = \sum_{S_1} a_{n_k} x_{m_k} + \sum_{S_2} a_{n_k} x_{m_k}$, and so either $\sum_{S_1} a_{n_k} x_{m_k} = \infty$ or $\sum_{S_2} a_{n_k} x_{m_k} = \infty$. *Choosing* S_1 *or* S_2 accordingly we produce the sequence (n_k,m_k) with the desired properties, (i.e. satisfying a) and b) in Theorem 1 and also satisfying $n_k \neq m_j$ for all k,j and $n_k > m_k$ for every k.).

Now consider the series $\sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$. *Since* $n_k > m_k$, *we have* $0 \le a_{n_k} - a_{m_k} \le a_{n_k}$ and $0 \le x_{m_k} - x_{n_k} \le x_{m_k}$, and so $0 \le (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$ $\le a_{n_k} x_{m_k} \to 0$ as $k \to \infty$. Furthermore, since $\sum_k a_{n_k} x_{m_k} = \infty$, $a_{m_k} x_{n_k} \ge 0$, $\sum_k a_{n_k} x_{n_k}$ $\le a \cdot x < \infty$, $\sum_k a_{m_k} x_{m_k} \le a \cdot x < \infty$, and $\sum a_{m_k} x_{n_k} \le \sum a_{n_k} x_{n_k} < \infty$, we have

$$\sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) = \sum_k (a_{n_k} x_{m_k} + a_{m_k} x_{n_k} - a_{n_k} x_{n_k} - a_{m_k} x_{m_k})$$

$$= \infty .$$

We shall now show that for every $\varepsilon > 0$, there exists a subsequence (k_n) of positive integers such that $\varepsilon = \sum_{k \in \{k_n\}} (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$. This follows from the following more general fact.

Suppose $(d(k))$ is a non-negative sequence for which $d(k) \to 0$ as $k \to \infty$ an $\sum d(k) = \infty$. We assert that very every $\varepsilon > 0$, there exists a subsequence (k_n) suc that $\varepsilon = \sum d(k_n)$. The proof of this fact proceeds along the same lines as the proof of Riemann's theorem on rearrangments of conditionally convergent series. Fix

$\varepsilon > 0$ and choose $n_1 \geq N_1$ so that $d(k) < \varepsilon$ for every $k \geq N_1$, and so that n_1 is the greatest integer greater than N_1 such that $\sum_{k=N_1}^{n_1} d(k) < \varepsilon$. Hence $\sum_{k=N_1}^{n_1} d(k) < \varepsilon \leq \sum_{k=N_1}^{n_1+1} d(k)$. This can be done since $d(k) \to 0$ as $k \to \infty$ and $d(k) = \infty$. Choose $N_2 > n_1$ so that $d(k) < (\varepsilon - \sum_{k=N_1}^{n_1} d(k))/2$ for every $k \geq N_2$ and then choose n_2 to be the largest integer greater than N_2 such that $\sum_{k=N_2}^{n_2} d(k) < \varepsilon - \sum_{k=N_1}^{n_1} d(k)$. Hence $\sum_{k=N_2}^{n_2} d(k) < \varepsilon - \sum_{k=N_1}^{n_1} d(k) \leq \sum_{k=N_2}^{n_2+1} d(k)$. Proceeding inductively in this way, we obtain sequences (N_p) and (n_p) of positive integers for which $n_p \geq N_p > n_{p-1}$, $0 \leq d(k) \leq (\varepsilon - \sum_{q=1}^{p-1} \sum_{k=N_q}^{n_q} d(k))/2^{p-1}$ for every p and every $k \geq N_p$, and

$$\sum_{k=N_p}^{n_p} d(k) < \varepsilon - \sum_{q=1}^{p-1} \sum_{k=N_q}^{n_q} d(k) \leq \sum_{k=N_p}^{n_p+1} d(k) .$$

This implies that

$$0 < \varepsilon - \sum_{q=1}^{p} \sum_{k=N_q}^{n_q} d(k) \leq d(n_p + 1) \leq (\varepsilon - \sum_{q=1}^{p-1} \sum_{k=N_q}^{n_q} d(k))/2^{p-1}$$

$$\leq \varepsilon/2^{p-1} \to 0 \quad \text{as} \quad p \to \infty.$$

Therefore $\varepsilon = \sum_{q=1}^{\infty} \sum_{k=N_q}^{n_q} d(k)$. Hence, if we choose (k_n) to be the strictly increasing sequence of positive integers k, where k is taken to range over the set $\bigcup_{p=1}^{\infty} \{k : N_p \leq k \leq n_p\}$, we have $\varepsilon = \sum d(k_n)$.

Applying this result to the sequence $(a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$, since it is non-negative, tends to 0, and sums to ∞, we obtain that for every $\varepsilon > 0$, there exist subsequences of (n_k) and (m_k), which we shall again denote by (n_k) and (m_k), for which $\varepsilon = \sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$.

Now recall that we wish to show that $S(a,x) = [a \cdot x, \infty]$. We already know $a \cdot x$ and $\infty \in S(a,x)$. Suppose $a \cdot x < r < \infty$. It suffices to show $r \in S(a,x)$. Let $\varepsilon = r - a \cdot x$ and choose subsequences which we again call (n_k) and (m_k) so that

$$\varepsilon = \sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) .$$

We now choose π, the requisite permutation on Z^+, as follows. Let $\pi(n_k) = m_k$ and $\pi(m_k) = n_k$ for each k, and let π fix all other integers n (i.e., those n for which $n \neq n_k, m_k$ for every k). The permutation π is well-defined since $n_i \neq m_j$ for every i,j. Let Z_π denote the set $\{n : n = n_k$ or $n = m_k$ for some $k\}$. Hence $\pi(n) = n$ for all $n \notin Z$. Then

$$\begin{aligned}\sum_n a_n x_{\pi(n)} &= \sum_{n \notin Z} a_n x_n + \sum_k (a_{n_k} x_{m_k} + a_{m_k} x_{n_k}) \\ &= \sum_{n \notin Z} a_n x_n + \sum_k (a_{n_k} x_{n_k} + a_{m_k} x_{m_k}) + (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})) \\ &= \sum a_n x_n + \sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) \\ &= a \cdot x + \varepsilon = r ,\end{aligned}$$

and so $r \in S(a,x)$, which proves (3). Q.E.D.

THEOREM 2. Let $a = (a_n)$ where $a_1 > 0$ and $a_n \uparrow \infty$. Then a_{n+1}/a_n is bounded if and only if, for every $x = (x_n)$ for which $x_n \overset{\downarrow}{\neq} 0$, $S(a,x) = [a \cdot x, \infty]$.

PROOF. If a_{n+1}/a_n is bounded, then by Theorem 1, if $x_n \overset{\downarrow}{\neq} 0$, then $x = (x_n)$ satisfies condition (2) of the theorem. Also by Theorem 1, since $a_n \uparrow \infty$ and $a_1 >$ condition (3) of the theorem is satisfied by x. That is, $S(a,x) = [a \cdot x, \infty]$.

Conversely, if $S(a,x) = [a \cdot x, \infty]$ for every $x = (x_n)$ for which $x_n \overset{\downarrow}{\neq} 0$, we claim that a_{n+1}/a_n must remain bounded.

Suppose to the contrary that a_{n+1}/a_n is not bounded. Let $h(n)$ denote the least positive integer k for which $k \geq n$ and $a_{k+1}/a_k \geq 4^n$. Clearly $h(n)$ is a non-decreasing function of n. Define $x_n = (a_{h(n)} 3^n)^{-1}$. Then $x_n \overset{\downarrow}{\neq} 0$. Letting $x = (x_n)$, we claim that $S(a,x) \neq [a \cdot x, \infty]$. In fact, we claim that $a \cdot x < 1$ but $1 \notin S(a,x)$. Indeed, $a \cdot x = \sum a_n x_n = \sum a_n (a_{h(n)} 3^n)^{-1} \leq \sum 3^{-n} = 1/2 < 1$. Furthermore letting π be any permuation of Z^+, if $\pi^{-1}(k) > h(k)$ for some k, then

$$\begin{aligned}\sum a_n x_{\pi(n)} \geq a_{\pi^{-1}(k)} x_k &\geq a_{h(k)+1} x_k = a_{h(k)+1} (a_{h(k)} 3^k)^{-1} \\ &\geq 4^k 3^{-k} > 1 .\end{aligned}$$

On the other hand, if $\pi^{-1}(k) \leq h(k)$ for every k, then

$$\sum a_n x_{\pi(n)} = \sum a_{\pi^{-1}(n)} x_n \leq a_{h(n)} x_n = \sum 3^{-n} = 1/2 < 1 .$$

In any case, $\sum a_n x_{\pi(n)} \neq 1$, hence $1 \notin S(a,x)$. Q.E.D.

NOTE. In the proof of Theorem 1, each time we constructed a permutation π to solve the equation $\sum a_n x_{\pi(n)} = r$, it sufficed to use only disjoint 2-cycles. That is, each such π that we constructed was the product of disjoint 2-cycles. This seems odd and leads us to ask if there are any circumstances in which the use of infinite-cycles or n-cycles yields more. In other words, is it always true that $S(a,x)$ is the same as $\{\sum a_n x_{\pi(n)} : \pi$ is a permutation of Z^+ which is a product of disjoint 2-cycles$\}$?

The following question seems likely to have an affirmative answer. If so, this would give a characterization for those sequences a and x where $a_n \uparrow \infty$, $a_1 > 0$, and $x_n \underset{\neq}{\downarrow} 0$, which satisfy $S(a,x) = [a \cdot x, \infty]$. However, it remains unsolved.

QUESTION 1. If a and x are as above, does (3) $\Longrightarrow$ (2) in Theorem 1?

Finally, we wish to point out that Theorems 1 and 2 imply analogous theorems in which a and x switch roles. Indeed, the proofs of the following two corollaries follow naturally along the same lines as those of Theorems 1 and 2.

COROLLARY 3. Let $x = (x_n)$ where $x_n > 0$ for all n, and $x_n \to 0$ as $n \to \infty$. Consider the following conditions.

(1) x_n/x_{n+1} is bounded below.

(2) For the non-negative sequence $a = (a_n)$, there exist subsequences (a_{n_k}) and (x_{m_k}) of a and x, respectively, such that

a) $a_{n_k} x_{m_k} \to 0$ as $k \to \infty$, and

b) $\sum_k a_{n_k} x_{m_k} = \infty$.

Then (1) implies that (2) holds for every strictly positive sequence $a = (a_n)$ that tends to ∞ .

COROLLARY 4. Let $x = (x_n)$ be a non-negative sequence. Then x_n/x_{n+1} is bounded below if and only if, for every $a = (a_n)$ for which $a_n \uparrow \infty$ and $a_1 > 0$, $S(a,x) = [a \cdot x, \infty]$.

QUESTION 2. Is there anything to be said about the qualitative nature of $S(a,x)$? Is it always a Borel set, measurable, F_σ, G_σ ?

REFERENCE

1. WEISS, GARY "Commutators and operator ideals", dissertation, University of Michigan, 1975.