

ON ALMOST BIPARTITE LARGE CHROMATIC GRAPHS

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Dedicated to Professor A. Kotzig on the occasion of his sixtieth birthday

0. Introduction

In the past we have published quite a few papers on chromatic numbers of graphs (finite or infinite), we give a list of those which are relevant to our present subject in the references. In this paper we will mainly deal with problems of the following type: Assuming that the chromatic number $\chi(\mathcal{G})$ of a graph \mathcal{G} is greater than κ , a finite or infinite cardinal, what can be said about the behaviour of the set of all finite subgraphs of \mathcal{G} . We will investigate this problem in case some other restrictions are imposed on \mathcal{G} as well. Most of the problems seem difficult and our results will give just some orientation. The results show that $\chi(\mathcal{G})$ can be arbitrarily large while the finite subgraphs are very close to bipartite graphs. It is clear from what was said above that this topic is a strange mixture of finite combinatorics and set theory and we recommend it only for those who are interested in both subjects. Finally we want to remind the reader the most striking difference between large chromatic finite and infinite graphs which was discovered by the first two authors about fifteen years ago [4]. While for any $k < \omega$ there are finite graphs with $\chi(\mathcal{G}) > k$ without any short circuits, [1], a graph with $\chi(\mathcal{G}) > \kappa \geq \omega$ has to contain a complete bipartite graph $[k, \kappa^+]$ for all $k < \omega$. Hence such a graph contains all finite bipartite graphs, though it may avoid short odd circuits.

Our set-theoretical notation will be standard as for graph theory we will use the notation of our joint paper with F. Galvin [3] with some self-explanatory changes.

1. The standard examples and the ordered edge graph

Definition 1.1. For each ordinal α , for $2 \leq k < \omega$, and for $1 \leq i \leq k-1$

$$\mathcal{G}_0(\alpha, k, i) = \langle V_0(\alpha, k, i), E(\alpha, k, i) \rangle$$

is the graph with $V_0(\alpha, k, i) = [\alpha]^k$, where $E(\alpha, k, i)$ is defined by the following stipulations. For $X \in [\alpha]^k$ we write $X = \{x_0, \dots, x_{k-1}\}$ with $x_0 < \dots < x_{k-1}$ and for $X, Y \in [\alpha]^k$

$$\{X, Y\} \in E(\alpha, k, i) \quad \text{iff} \quad y_j = x_{i+j} \text{ for } j < k-i-1.$$

We call $\mathcal{G}_0(\alpha, k, 1) = \mathcal{G}_0(\alpha, k)$ and we call this graph the k -edge graph on α .

Definition 1.2. For each ordinal α , for $3 \leq k < \omega$ and for $1 \leq i < k - 1$,

$$\mathcal{G}_1(\alpha, k, i) = \langle V_1(\alpha, k, i), E_1(\alpha, k, i) \rangle$$

is the graph with $V_1(\alpha, k, i) = [\alpha]^k$ where for $X, Y \in [\alpha]^k$

$$X, Y \in E_1(\alpha, k, i) \text{ iff } x_i < y_0 < x_{i+1} < \dots < x_{k-2} < y_{k-i-2} < x_{k-2} < y_{k-i-1}.$$

We call $\mathcal{G}_1(\alpha, k, i)$ the k, i -Specker graph on α , and just as in the case above we omit i for $i = 1$.

The above graphs are standard examples of large chromatic graphs, we list some properties of them.

Lemma 1.1. (For the proofs see [3] and [4]).

(a) $\chi(\mathcal{G}_0(\alpha, k, i)) > \kappa$ provided $\alpha \rightarrow (2k)_\kappa^k$ and, as a corollary of this, for all $\kappa \geq \omega$, $\chi(\mathcal{G}_0(\alpha, k, i)) > \kappa$ provided

$$\alpha \geq (\exp_{k-1}(\kappa))^+ \text{ for all } 2 \leq k < \omega, 1 \leq i \leq k - 1.$$

(b) $\chi(\mathcal{G}_1(\kappa, k, i)) = \kappa$ for all $\kappa \geq \omega$, $3 \leq k < \omega$, $1 \leq i \leq k - 1$, and as a corollary of this

$$\chi(\mathcal{G}_1(n, k, i)) \rightarrow +\infty \text{ if } n \rightarrow +\infty.$$

(c) $\mathcal{G}_0(\alpha, k)$ does not contain odd circuits of length $2j + 1$ for $1 \leq j \leq k - 1$.

(d) $\mathcal{G}_1(\alpha, k)$ does not contain a complete κ -graph for $3 \leq k < \omega$, and $\mathcal{G}_1(\alpha, n^2 + n + 1, n)$ does not contain odd circuits of length $2j + 1$ for $1 \leq j \leq n$, $1 \leq n < \omega$.

Definition 1.3. For a graph $\mathcal{G} = \langle V, E \rangle$ and for an ordering \prec of V we define the ordered edge graph $\text{OE}(\mathcal{G}, \prec) = \langle V(\mathcal{G}, \prec), E(\mathcal{G}, \prec) \rangle$ of \mathcal{G} for the ordering \prec , as follows $V(\mathcal{G}, \prec) = E$. For $X \in [V]^2$ write $X = \{x_0, x_1\}$ where $x_0 \prec x_1$. For $X, Y \in E$ put $\{X, Y\} \in E(\mathcal{G}, \prec)$ iff either

$$x_1 = y_0 \text{ or } y_1 = x_0.$$

It is clear from the definition that the 2-edge graph $\mathcal{G}_0(\alpha, 2)$ is the ordered edge graph of the complete graph with vertex set α for the natural ordering of α .

More generally the following is true:

Lemma 1.2. For each α , and for $1 \leq k < \omega$ there is a well-ordering $\prec_{\alpha, k}$ of $[\alpha]^k$ such that $\mathcal{G}_0(\alpha, k + 1)$ is the ordered edge graph of $\mathcal{G}_0(\alpha, k)$ for the ordering $\prec_{\alpha, k}$. Here $\mathcal{G}_0(\alpha, 1)$ is the complete graph on α -vertices and $\prec_{\alpha, 1}$ is the natural ordering of α .

Proof. $\prec_{\alpha, k}$ can be any ordering of $[\alpha]^k$ satisfying

$$\min X < \min Y \Rightarrow X \prec_{\alpha, k} Y.$$

Lemma 1.3. Let \mathcal{G} be a graph with $\chi(\mathcal{G}) \leq 2^\kappa$ for $\kappa \geq 1$ and let \prec be any ordering of the vertices. Then $\chi(\text{OE}(\mathcal{G}, \prec)) \leq 2\kappa$.

Proof. Let \mathcal{G} be $\mathcal{G} = \langle V, E \rangle$. Then V is the union of 2^κ independent sets. Since the

complete graph of 2^κ vertices is the union of κ bipartite graphs, there are disjoint partitions $V = A_i \cup B_i$ of V for $i < \kappa$ such that $E \subset \bigcup_{i < \kappa} [A_i, B_i]$. We now partition each $[A_i, B_i]$ into the union of two sets independent in $OE(\mathcal{G})$. Let $C_j^i = \{\{x_0, x_1\} \in [A_i, B_i] : x_j \in A_i\}$ for $j < 2$. Clearly C_j^i contains no edge of $OE(\mathcal{G})$ and $C_0^i \cup C_1^i = [A_i, B_i]$ for $i < \kappa$.

In what follows \log denotes the 2 basis logarithm and $\log^{(k)}$ is the k -times iterated logarithm function.

Corollary 1.4. *If \mathcal{G} is a subgraph of n vertices of some $\mathcal{G}_0(\alpha, k)$ for $k \geq 2$ then*

$$\chi(\mathcal{G}) \leq c_k \log^{(k-1)}(n) \quad \text{for some } c_k > 0.$$

To close this section we mention the first problem we cannot solve.

Definition 1.4. For each (infinite) graph $\mathcal{G} = \langle V, E \rangle$ let

$$f_{\mathcal{G}}^0(n) = \max\{\chi(\mathcal{G}(A)) : A \subset V \wedge |A| = n\}.$$

Problem 1. For what functions $f : \omega \rightarrow \omega$ is it true that for all cardinals $\kappa > \omega$ there is a graph \mathcal{G} with $\chi(\mathcal{G}) > \kappa$ and $f_{\mathcal{G}}^0(n) \leq f(n)$ for $n < \omega$.

Corollary 1.4 shows that $\log^{(k)}(n)$ is such a function for all $k < \omega$ and it is clear that $f_{\mathcal{G}}^0(n) \rightarrow +\infty$ for $n \rightarrow +\infty$ for all graphs with $\chi(\mathcal{G}) \geq \omega$. In [5] we ventured a conjecture that if \mathcal{F} is a class of finite graphs such that for all $\kappa > \omega$ there is a graph with $\chi(\mathcal{G}) > \kappa$ all whose finite subgraphs are in \mathcal{F} , then \mathcal{F} must contain all finite subgraphs of $\mathcal{G}_0(\omega, k)$ for some $k < \omega$. This, if true, would imply that the result given by Corollary 1.4 is best possible, but we do not really believe in this. It should be noted that a theorem of Erdős [2, p. 172] implies that for all functions f with $f(n) \rightarrow +\infty$ for $n \rightarrow \infty$ there exists a graph \mathcal{G} with $\chi(\mathcal{G}) = \omega$ and $f_{\mathcal{G}}^0(n) \leq f(n)$.

Finally we mention that the problem of the order of magnitude of $f_{\mathcal{G}}^0(n)$ for large κ -chromatic graphs of cardinality κ is completely open. We do not know $f_{\mathcal{G}}^0(n)$ for $\mathcal{G} = \mathcal{G}_1(\omega, 3)$.

2. Omitting vertices of subgraphs

Definition 2.1. Let $\mathcal{G} = \langle V, E \rangle$ be a graph;

$$f_{\mathcal{G}}^1(n) = \min\{\max\{|Z| : Z \subset A \wedge Z \text{ is independent}\} : A \subset V \wedge |A| = n\},$$

$$f_{\mathcal{G}}^2(n) = \min\{\max\{|Z| : Z \subset A \wedge \mathcal{G}(Z) \text{ is bipartite}\} : A \subset V \wedge |A| = n\}.$$

Clearly $f_{\mathcal{G}}^1(n) \geq \frac{1}{2} f_{\mathcal{G}}^2(n)$ holds. We are interested in the problem if these functions can be large for a graph of large chromatic number. The following theorem contains our main information about this.

Theorem 1. For all $\varepsilon > 0$ and for all κ there is a graph \mathcal{G} with $\chi(\mathcal{G}) > \kappa$ such that

$$f_{\mathcal{G}}^2(n) \geq (1 - \varepsilon)n$$

holds for all $n < \omega$.

Proof. By Lemma 1.1(a) we only have to show that for $\mathcal{G}_k = \mathcal{G}_0(\omega, k)$ $f_{\mathcal{G}_k}^2(n) \geq (1 - 2/k)n$. Let $A \subset [\omega]^k$, $|A| = n$, for some $2 \leq k < \omega$ and for some n . We prove, by induction on n , that there is a set $Z \subset A$, $|Z| \geq (1 - 1/2k)n$ such that Z spans a bipartite graph of $\mathcal{G}(\omega, k)$. The statement is trivial for $n = 1$. Assume $n > 1$ and that the statement is true for all $n' < n$. For $x \in \omega$ let $S(x) = \{X \in A : x \in X\}$, and $P(x) = \{X \in A : x = x_i \text{ for } 1 \leq i \leq k - 2 \text{ where } X = \{x_0, \dots, x_{k-1}\}\}$. By averaging, there is an $x \in \omega$, with $|S(x)| > 0$ and $|P(x)| \geq (1 - 2/k)|S(x)|$. Now the graph induced by $P(x)$ can be shown to be bipartite by the following partition: $A_0 = \{X \in P(x) : x = x_i \wedge i \text{ is even}\}$, $A_1 = \{X \in P(x) : x = x_i \wedge i \text{ is odd}\}$. $A_0 \cup A_1 = P(x)$ and A_j contains no edge of $\mathcal{G}(\omega, k)$ for $j < 2$. By induction, there is a subset $Q \subset A \setminus S(x)$, $|Q| \geq (1 - 2/k)(n - |S(x)|)$, which induces a bipartite graph of $\mathcal{G}_0(\omega, k)$. Since no edge of $\mathcal{G}_0(\omega, k)$ joins a point of $P(x)$ to a point of $A \setminus S(x)$, $Z = P(x) \cup Q$ satisfies the requirements of the theorem.

The theorem yields that there is a sequence $\varepsilon_n \rightarrow 0$ so that for some \mathcal{G} with $\chi(\mathcal{G}) = \omega$, $f_{\mathcal{G}}^2(n) \geq n(1 - \varepsilon_n)$. We do not know how fast ε_n can tend to 0. A result of Folkman gives information for the case $\frac{1}{2}n - f_{\mathcal{G}}^1(n)$ is bounded. This says that if $\frac{1}{2}n - f_{\mathcal{G}}^1(n) \leq k$ then $\chi(\mathcal{G}) \leq 2k + 2$.

Here we at least know that the situation is different for graphs with chromatic number $> \omega$.

Lemma 2.1. If $\chi(\mathcal{G}) > \omega$ then there is an $\varepsilon > 0$ such that $f_{\mathcal{G}}^1(n) \leq (\frac{1}{2} - \varepsilon)n$.

Proof. Clearly \mathcal{G} contains \aleph_1 vertex disjoint copies of some C_{2i+1} for some $i < \omega$. The union of m copies has cardinality $m(2i + 1)$ and contains no free set larger than mi .

Problem 2. Does there exist a graph \mathcal{G} and $c > 0$ such that $\chi(\mathcal{G}) = \omega_1$, $|\mathcal{G}| = \omega_1$ and $f_{\mathcal{G}}^1(n) \geq cn$.

Here we only have the very little information that the Specker graph $\mathcal{G}_1(\omega_1, 3)$ does not have this property. To see this it is sufficient to prove:

Theorem 2. Let $\mathcal{G} = \mathcal{G}_1(\omega, 3)$. Then

$$f_{\mathcal{G}}^1(m) \leq O\left(\frac{m \log \log m}{\log m}\right).$$

Proof. We define an $\mathcal{F} \subset [n]^3$ with $|\mathcal{F}| \geq n \log n/8 \log \log n$ such that for all $M \subset \mathcal{F}$, M independent in \mathcal{G} , $|M| \leq 4n$ holds for all sufficiently large n . This will prove the result. Let $a_i = \lceil \log n^i \rceil$ for $i < i_0 = \lceil \log n / \log \log n \rceil$. Then $a_i \in n$ for $i < i_0$. Let $X(t, i) = \{t, t + a_{2i}, t + a_{2i+1}\}$ and $\mathcal{F} = \{X(t, i) : X(t, i) \subset n\}$. Then $|\mathcal{F}| \geq n \log n/8 \log \log n$ provided n is large enough. Let $M \subset \mathcal{F}$, $|M| \geq 4n$. We prove that M contains an edge of

\mathcal{G} . Let $M_i = \{t : X(t, i) \in M\}$. Then $\sum_{i < i_0} |M_i| = |M| \geq 4n$. Let

$$N_i = \{t \in M_i : \exists t' \in M_i \ t' + a_{2i} < t < t' + a_{2i+1}\}.$$

We claim that $|N_i| \geq |M_i| - 2n/\log n$. We can choose elements t_0, \dots, t_{j-1} of M_i so that $M_i \subset \bigcup \{[t_l, t_l + a_{2i+1}) : l < j\}$ where the above intervals are pairwise disjoint. Now

$$M_i \setminus N_i \subset \bigcup \{[t_l, t_l + a_{2i+1}) \setminus (t_l + a_{2i}, t_l + a_{2i+1}) : l < j\}.$$

Since

$$\frac{a_{2i}}{a_{2i+1}} < \frac{2}{\log n},$$

$$|M_i - N_i| \leq 2n/\log n.$$

It follows now, that

$$\sum_{i < i_0} |N_i| \geq \sum_{i < i_0} |M_i| - \frac{2n}{\log \log n} \geq 4n - \frac{2n}{\log \log n} > n.$$

Then there are $i_1 < i_2$ with $N_{i_1} \cap N_{i_2} \neq \emptyset$. Let $t \in N_{i_1} \cap N_{i_2}$. Then for some $t' \in M_{i_1}$, $t' + a_{2i_1} < t < t' + a_{2i_1+1}$, $X(t', i_1) \in M$. On the other hand $X(t, i_2) \in M_{i_2} \subset M$. Now $t + a_{2i_2} > t' + a_{2i_2} > t' + a_{2i_1+1}$ since $\lceil \log n^{2i_1+1} \rceil < \lceil \log n^{2i_2} \rceil$ if $i_1 < i_2$ and n is large enough. Then $X(t', i_1)$ and $X(t, i_2)$ are adjacent in \mathcal{G} .

3. Omitting edges of a subgraph

Definition 3.1. Let

$$f_{\mathcal{G}}^3(n) = \max \{ \min \{ |E'| : \langle A, [A]^2 \cap \mathcal{G} \setminus E' \rangle \text{ is bipartite} \} : A \subset V \wedge |A| = n \};$$

more generally for $2 \leq k < \omega$,

$$f_{\mathcal{G}}^3(n, k) = \max \{ \min \{ |E'| : \langle A, [A]^2 \cap \mathcal{G} \setminus E' \rangle \text{ has chromatic number } \leq k \} : A \subset V \wedge |A| = n \}.$$

Clearly $f_{\mathcal{G}}^3(n, 2) = f_{\mathcal{G}}^3(n)$, and it is obvious that

$$f_{\mathcal{G}}^3(n, k) \leq f_{\mathcal{G}}^3(n, k') \quad \text{for } k' < k.$$

It is also clear from the definition that $n - f_{\mathcal{G}}^2(n) \leq 2f_{\mathcal{G}}^3(n)$. This in view of Lemma 2.1 implies that for any given graph \mathcal{G} with $\chi(\mathcal{G}) > \omega$ there is an $\varepsilon > 0$ such that $f_{\mathcal{G}}^3(n) \geq \varepsilon n$.

This contrasts again with the situation for finite graphs. L. Lovász recently informed us that, as a generalization of a theorem of T. Gallai, he can prove the following theorem: *For $2 \leq r < \omega$ there is a finite graph \mathcal{G} with $\chi(\mathcal{G}) \geq r + 2$ and $f_{\mathcal{G}}^3(n) = O(n^{1-1/r})$.*

We describe his example: Let m be even. Let V be the set of r -dimensional lattice points mod (m) , $(x_0, \dots, x_{r-1}), (y_0, \dots, y_{r-1})$ are connected if for some $i < r$ $|x_i - y_i| = 1$, and $x_j = y_j$ for $j \neq i, j < r$. This graph satisfies the requirements for large enough m . In case of infinite chromatic graphs we are again left with the examples $\mathcal{G}_0(\alpha, k)$.

Theorem 3. (a) For all $\kappa \geq \omega$ there exists a graph with

$$\chi(\mathcal{G}) > \kappa \quad \text{and} \quad f_{\mathcal{G}}^3(n) \leq 2n^{3/2}$$

(b) $\forall \varepsilon > 0$ there is an $r < \omega$ such that for all $\kappa \geq \omega$ there exists a graph with $\chi(\mathcal{G}) > \kappa$ and $f_{\mathcal{G}}^3(n, r) \leq n^{1+\varepsilon}$.

By Lemma 1.1 it is sufficient to prove

Theorem 3.A. (a) If $\mathcal{G} = \mathcal{G}_0(\omega, 2)$, then $f_{\mathcal{G}}^3(n) \leq 2n^{3/2}$.

(b) If $\mathcal{G} = \mathcal{G}(\omega, k)$ for some $3 \leq k < \omega$, then for all $\eta > 0$ there is an $r < \omega$ such that $f_{\mathcal{G}}^3(\omega, r) \leq n^{1+k-1+\eta}$.

Proof. (a) Let $V \subset [\omega]^2$, $|V| = n$. Put $V(x) = \{\{x, y\} : \{x, y\} \in V\}$, $v(x) = |V(x)|$ for $x \in \omega$. For $e \in V$, let $\mathcal{G}(e, V) = \{e' \in V : \{e, e'\} \in \mathcal{G}\}$. Let $A = \{x \in \omega : v(x) \geq n^{1/2}\}$. Then $|A| \leq n^{1/2}$, since $|V| = n$. We now split V into the union of three disjoint sets. Let $V_j = \{e \in V : |e \cap A| = j\}$ for $j < 3$.

We claim that for all $e \in V$ $|\mathcal{G}(e, V_0 \cup V_2)| \leq 2n^{1/2}$. Indeed let $e = \{x, y\}$. Then

$$|\mathcal{G}(e, V_0 \cup V_2)| \leq |\{\{u, x\} : \{u, x\} \in V_0 \cup V_2\}| + |\{\{u, y\} : \{u, y\} \in V_0 \cup V_2\}|.$$

Clearly it is sufficient to see that $|\{\{u, x\} : \{u, x\} \in V_0 \cup V_2\}| \leq n^{1/2}$. For $x \in A$ this holds because $|A| \leq n^{1/2}$, for $x \notin A$ this holds because $v(x) \leq n^{1/2}$. It follows that the $|\bigcup \{\mathcal{G}(e, V_0 \cup V_2) : e \in V\}| \leq 2n^{3/2}$. Omitting all these edges from the subgraph spanned by V , only the edges spanned by V_1 remain there. But $\mathcal{G}(V_1)$ is bipartite as shown by the partition $\tilde{V}_1 = \{e \in V_1 : \min e \in A\}$, $\tilde{V}_2 = \{e \in V_1 : \max e \in A\}$.

(b) By (a) and by Lemma 1.2 this follows, by induction on k , $2 \leq k < \omega$, from the following theorem:

Theorem 4. Let $2 \leq k < \omega$. \mathcal{G} is said to have property $P(k)$ if for all $\eta > 0$ there is an $r < \omega$ with $f_{\mathcal{G}}^3(n, r) < n^{1+k-1+\eta}$. If $\mathcal{G} = \langle V, E \rangle$ has property $P(k)$ and \prec is any ordering of V , then $\text{OE}(\mathcal{G}, \prec)$ has the property $P(k+1)$.

Proof. Let $r(\eta) = \min\{r : f_{\mathcal{G}}^3(n, r) < n^{1+k-1+\eta} \text{ holds for all } n < \omega\}$. We now have to show that for all η there is an s such that for all $n < \omega$

$$f_{\text{OE}(\mathcal{G})}^3(n, s) < n^{1+(k+1)-1+\eta}.$$

Given $\eta > 0$, first choose $\eta' < \eta$, $\eta' > 0$. Then choose $\eta'' > 0$ so that $(1 - (k+1)^{-1} - \eta') \cdot (1 + k^{-1} + \eta'') = 1 - \delta$ for some $\delta > 0$. Let l be an integer with $1 \leq l\delta$. We claim that

$$\max\{(r(\eta'') + 2)l, \min\{n : 2l n^{1+(k+1)-1+\eta'} < n^{1+(k+1)-1+\eta}\}\} = s$$

satisfies the requirements of the theorem. Let $E' \subset E$, $|E'| = n$. For $x \in V$ let $E'(x) = \{\{x, y\} : \{x, y\} \in E'\}$ and $e'(x) = |E'(x)|$. For $E'' \subset E'$ and $e \in E'$

$$\mathcal{G}(e, E'') = \{e' \in E'' : \{e, e'\} \in \text{OE}(\mathcal{G}, \prec)\}.$$

Let $A = \{x \in V : e'(x) \geq n^{(k+1)^{-1} + \eta'}\}$. Then $|A| \leq n^{1-(k+1)^{-1} - \eta'}$. We partition the set E' into three pieces. Let $E_j = \{e \in E' : |e - A| = j\}$ for $j < 3$. For each $e \in E'$, $|\mathcal{G}(e, E_2)| \leq 2n^{(k+1)^{-1} + \eta'}$, hence omitting $2n^{1+(k+1)^{-1} + \eta'}$ edges. E_2 is independent and no edge

joins any point of E_2 to any other point of E' . The subgraph spanned by E_1 in the ordered edge graph is clearly 2-chromatic, $E_0 \subseteq [A]^2$. Since $|A| \leq n^{1-(r+1)^{-1}-\eta'}$, by the assumption, there is an $E^* \subset E_0$, $|E^*| \leq |A|^{1+k^{-1}+\eta'} = n^{1-\delta}$ so that $\langle A, E^0 \setminus E^* \rangle$ has chromatic number at most $r(\eta'')$. But then, by Lemma 1.3 the subgraph induced by $E_0 \setminus E^*$ in the ordered edge graph has chromatic number $\leq r(\eta'')$ too. We now proved that omitting $2n^{1+(r+1)^{-1}+\eta'}$ edges of the ordered edge graph there is $E^* \subset E'$ so that the subgraph spanned by $E' \setminus E^*$ is at most $2+r(\eta'')$ chromatic, and $|E^*| \leq n^{1-\delta}$. Repeating this procedure at most l times we obtain that omitting $2ln^{1+(r+1)^{-1}+\eta'}$ ($< n^{1+(r+1)^{-1}+\eta'}$ for $n \geq s$) edges the ordered edge graph is at most s chromatic.

We think that the main unsolved problem of this section is of finite character. It is not known if Lovász' example is best possible.

Problem 3. Assume \mathcal{G} has chromatic number ω . Can $f_{\mathcal{G}}^3(n)$ tend to infinity very slowly? Can it be at most $\log n$ or $\log^{(k)}(n)$ for $k < \omega$?

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