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ON A PROBLEM IN COMBINATORIAL GEOMETRY

P. ERDÖS, G. PURDY and E.G. STRAUS*
Dept. of Mathematics, Texas A & M University, College Station, TX 77843, USA

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1. Introduction

Let S be a set of n points in the plane not all on one straight line. Let T be the maximal area and t the minimal area of nondegenerate triangles with all vertices in S. Let $f(S)=T/t$ and $f(n)=\inf_s f(S)$.

In this note we prove that

(1.1) $\quad f(n)=[\frac{1}{2}(n-1)]$

for all sufficiently large n (we conjecture that (1.1) holds for all $n>5$). It is known [1] that $f(5)=\frac{1}{2}(\sqrt{5}+1)$, attained in case 5 is the set of vertices of a regular pentagon.

The fact that $f(n)\leq[\frac{1}{2}(n-1)]$ can be verified by considering the set

$$S_0=\{(0,0),(1,0),\ldots,([\tfrac{1}{2}(n-1)],0),(0,1),(1,1),\ldots,([\tfrac{1}{2}(n-2)],1)\}$$

of equally spaced points on two parallel lines.

We shall use the notation $\mathscr{C}(S)$ for the convex hull of S, $\mathscr{T}(\mathscr{C})$ for a triangle of maximal area contained in the convex set $\mathscr{C}$ and $|X|$ for the area of the convex set X.

In Section 2 we state and prove our main result. In Section 3 we give some related problems and conjectures.

We need the following result about extremal values of $|\mathscr{C}|/|\mathscr{T}(\mathscr{C})|$.

1.2. Theorem. *For all convex regions $\mathscr{C}$ we have*

$$|\mathscr{C}|/|\mathscr{T}(\mathscr{C})|\leq\frac{4\pi}{3\sqrt{3}}<2.4184.$$

The maximum is attained if and only if $\mathscr{C}$ is elliptic.

Proof. See [2].

Finally we need a result about the triangulation of polygons.

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1.3. Theorem. *Let S be a set of n points in the plane not all on one straight line. If there are k points of S on the boundary of the convex hull $\mathscr{C}(S)$ and $n-k$ in the interior of $\mathscr{C}(S)$, then any triangulation of $\mathscr{C}(S)$ whose vertices are all the points of S contains $2n-k-2$ triangles.*

Proof. Obvious by induction on n.

2. Evaluation of $f(n)$

In this section we prove our main result.

2.1. Theorem. *If $n>37$, then*

$$f(n)=[\tfrac{1}{2}(n-1)].$$

If n is even and $n\geq 38$, then any set S with $f(S)=f(n)$ is affine equivalent to the set S_0 of the introduction.

The proof is via a sequence of lemmas.

2.2. Lemma. *If $f(S)=f(n)$ and S has k points on the boundary of $\mathscr{C}(S)$, then*

$$k>\left(2-\frac{2\pi}{3\sqrt{3}}\right)(n-1)>0.7908\ (n-1).$$

Proof. By Theorem 1.3 we have

$$|\mathscr{C}|\geq(2(n-1)-k)t \tag{2.3}$$

and by Theorem 1.2 we have

$$|\mathscr{C}|<\frac{4\pi}{3\sqrt{3}}T\leq\frac{4\pi}{3\sqrt{3}}\left[\frac{n-1}{2}\right]t\leq\frac{2\pi}{3\sqrt{3}}(n-1)t. \tag{2.4}$$

The result now follows from (2.3) and (2.4).

2.5. Lemma. *If $f(S)=f(n)$ and $n>37$, then every maximal triangle $\mathscr{T}$ has one edge on the boundary of $\mathscr{C}(S)$.*

Proof. Assume that there exists a $\mathscr{T}$ with no edge on the boundary of $\mathscr{C}$ and triangulate the three portions of $\mathscr{C}$ which are exterior of $\mathscr{T}$ using the points of S on the boundary of $\mathscr{C}$. Assume that the three boundary arcs contain k_1, k_2 and k_3 points of S respectively. Then $k_1+k_2+k_3=k+3$. By Theorem 1.3 the triangulation of $\mathscr{C}\setminus\mathscr{T}$ yields $k_1+k_2+k_3-6=k-3$ triangles. Thus, by Lemma 2.2,

$$|\mathscr{C}-\mathscr{T}|\geq(k-3)t\geq\left(\left(2-\frac{2\pi}{3\sqrt{3}}\right)(n-1)-3\right)t.$$

On the other hand

$$|\mathscr{C}-\mathscr{T}|<\left(\frac{4\pi}{3\sqrt{3}}-1\right)T\leq\left(\frac{4\pi}{3\sqrt{3}}-1\right)\left[\frac{n-1}{2}\right]t.$$

Thus

$$\left(2-\frac{2\pi}{3\sqrt{3}}\right)(n-1)-3<\left(\frac{2\pi}{3\sqrt{3}}-\frac{1}{2}\right)(n-1)$$

or

$$n-1<6\Big/\left(5-\frac{8\pi}{3\sqrt{3}}\right)<36.7644,$$

that is $n\leq 37$.

2.6. Lemma. *If a convex region $\mathscr{C}$ contains a maximal triangle $\mathscr{T}$ with two sides on the boundary of $\mathscr{C}$, then $|\mathscr{C}|\leq 2|\mathscr{T}|$.*

Proof. Let $\mathscr{T}=\triangle ABC$ with sides AB and AC on the boundary of $\mathscr{C}$. Then through the vertex B there is a line of support l of $\mathscr{C}$ parallel to AC and through the vertex C there is a line of support l' of $\mathscr{C}$ parallel to AB. Let D be the point of intersection of l and l' then $\mathscr{C}$ is contained in the parallelogram $ABDC$ whose area is $2|\mathscr{T}|$.

2.7. Lemma. *If $f(S)=f(n)$ and $|\mathscr{C}(S)|\leq 2T$, then $f(n)=[\frac{1}{2}(n-1)]$.*

Proof. By Theorem 1.3 we have

$$(2(n-1)-k)t\leq|\mathscr{C}|\leq 2T\leq 2[\tfrac{1}{2}(n-1)]t$$

and hence

$$[\tfrac{1}{2}(n-1)]\geq T/t\geq n-1-\tfrac{1}{2}k\geq\tfrac{1}{2}n-1.$$

This proves the lemma in case n is even.

If n is odd pick a maximal triangle $\mathscr{T}$. If $\mathscr{T}$ contains at least $\frac{1}{2}(n+3)$ points of S, then a triangulation of $\mathscr{T}$ yields $T\geq\frac{1}{2}(n-1)t$ and we are finished. We may therefore assume that $\mathscr{T}$ contains $n_0\leq\frac{1}{2}(n+1)$ points of S. But then the closure of $\mathscr{C}\setminus\mathscr{T}$ contains at least $n-n_0+2=\frac{1}{2}(n+3)$ points and triangulation of $\mathscr{C}-\mathscr{T}$ gives at least $\frac{1}{2}(n-1)$ triangles. Thus

$$T\geq|\mathscr{C}\setminus\mathscr{T}|\geq\tfrac{1}{2}(n-1)t.$$

In view of Lemmas 2.5, 2.6 and 2.7 we may assume from now on that for all maximal triangles $\mathscr{T}$ the set $\mathscr{C}\setminus\mathscr{T}$ consists of two convex regions. By affine transformation we can normalize the situation so that one maximal triangle is an equilateral $\triangle ABC$ with side AB on the boundary of $\mathscr{C}$ (Fig. 1). By the maximality of $\triangle ABC$ we have that $\mathscr{C}$ is contained in the trapezoid $ABDE$, and by assumption we can choose ABC so that there are points of S in the interiors of

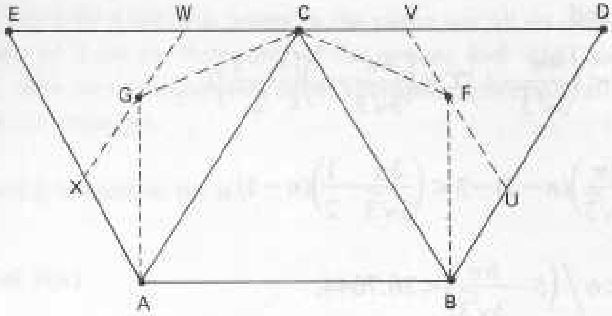

Fig. 1.

$\triangle BCD$ and $\triangle ACE$. Let F be the point of S in $\triangle BCD$ with maximal distances from BC and G the point of S in $\triangle ACE$ with maximal distance from AC. Then $\mathscr{C}$ is contained in the hexagon $ABUVWX$ where UV is the line through F parallel to BC and WX is the line through G parallel to AC.

Now $|\triangle BUF|+|\triangle CFV|$ is maximal when $|\triangle BCF|=\frac{1}{2}T$ and therefore, for $\mathscr{P}=ABFCG$, we have

$$|\mathscr{C}|-|\mathscr{P}|\leq|ABUVWX|-|\mathscr{P}|\leq\tfrac{1}{2}T\leq\tfrac{1}{4}(n-1)t.$$

Thus there cannot be more than $\frac{1}{4}(n-1)$ points of S exterior to $\mathscr{P}$ and hence there are

$$k_1\geq k-\tfrac{1}{4}(n-1)>0.5408(n-1) \tag{2.8}$$

points of S which are boundary points of $\mathscr{C}$ on the boundary of $\mathscr{P}$.

2.9. Lemma. *If an edge* $\mathscr{E}$ *of the boundary of* $\mathscr{C}$ *contains* $c(n-1)+1$ *points of* S *and* $f(S)=f(n)$, *then either all points of* $S\setminus\mathscr{E}$ *are collinear, or*

$$c<\frac{2\pi}{3\sqrt{3}}-1+\frac{2}{n-1}<0.2092+\frac{2}{n+1}.$$

Proof. Let the length of $\mathscr{E}$ be L. The shortest interval determined by points of S on $\mathscr{E}$ has length at most $L/c(n-1)$. Thus any point of $S\setminus\mathscr{E}$ has distance at least

$$h=\frac{2tc(n-1)}{L}\geq\frac{4c}{L}T$$

from $\mathscr{E}$. Since $|\mathscr{C}|>2T$ the two edges adjacent to $\mathscr{E}$ have sum of interior angles $>\pi$ with $\mathscr{E}$. Thus the part of $\mathscr{C}$ within a distance h of $\mathscr{E}$ has area greater than

$$hL\geq 4cT.$$

Thus, by Theorem 1.2, the convex hull $\mathscr{C}_1=\mathscr{C}(S\setminus\mathscr{E})$ has area less than

$$\left(\frac{4\pi}{3\sqrt{3}}-4c\right)T\leq\left(\frac{2\pi}{3\sqrt{3}}-2c\right)(n-1)t$$

and contains at least $(1-c)(n-1)$ points of S. If $|\mathscr{C}_1|>0$ then triangulation of $\mathscr{C}_1$ yields at least $(1-c)(n-1)-2$ triangles. Hence

$$\left(\frac{2\pi}{3\sqrt{3}}-2c\right)(n-1)t>(1-c)(n-1)t-2t$$

so that

$$c<\frac{2\pi}{3\sqrt{3}}-1+\frac{2}{n-1}<0.2092+\frac{2}{n-1}.$$

Comparing inequality (2.8) and Lemma 2.9 we see that there must be at least three edges of $\mathscr{P}$ on the boundary of $\mathscr{C}$. Moreover there must be two adjacent edges of $\mathscr{P}$ which together contain more than

$$(2.10)\quad 2+(0.5408-0.2092)(n-1)=2+0.3316(n-1)$$

point of S.

2.11. Lemma. *If two adjacent edges of $\mathscr{P}$ lie on the boundary of $\mathscr{C}$ and contain $2+c_1(n-1)$ and $2+c_2(n-1)$ points of S respectively, where $c_1\geq c_2>0$, then for $f(S)=f(n)$ and $n>37$ we have*

$$c_1+c_2<\tfrac{1}{4}+(c_1-c_2)^2.$$

Proof. Let $\mathscr{A}, \mathscr{B}$ be the two edges and V the common vertex. Let a, b be the lengths of $\mathscr{A}, \mathscr{B}$. Since by assumption no triangle of maximal area has two edges on the boundary of $\mathscr{C}$, it follows that the triangle $\mathscr{T}_0$ with sides $\mathscr{A}, \mathscr{B}$ has area $|\mathscr{T}_0|<T$. Let xa be the minimal distance from V to $(S\cap\mathscr{A})\setminus\mathscr{B}$ and let yb be the minimal distance from V to $(S\cap\mathscr{B})\setminus\mathscr{A}$. Then $\mathscr{A}$ contains an interval of length at most $(1-x)a/c_1(n-1)$ with endpoints in S. This interval, together with the nearest point to V of $(S\cap\mathscr{B})\setminus\mathscr{A}$ forms a triangle whose area is at most

$$\frac{y(1-x)}{c_1(n-1)}|\mathscr{T}_0|<\frac{y(1-x)}{c_1(n-1)}T\leq\frac{y(1-x)}{c_1(n-1)}\frac{n-1}{2}t=\frac{y(1-x)}{2c_1}t.$$

Thus we must have

$$(2.12)\quad 2c_1<y-xy.$$

In a completely analogous manner we get

$$(2.13)\quad 2c_2<x-xy.$$

Combining (2.12) and (2.13) we have

$$xy>(2c_1+xy)(2c_2+xy)$$

so

$$0\leq(xy+c_1+c_2-\tfrac{1}{2})^2<(c_1+c_2-\tfrac{1}{2})^2-4c_1c_2$$
$$=(c_1-c_2)^2-(c_1+c_2)+\tfrac{1}{4}$$

as was to be proved.

Now, by (2.10), we have

$$c_1+c_2>0.3316-\frac{1}{n-1},$$

$$c_1-c_2=2c_1-(c_1+c_2)<0.4184+\frac{4}{n-1}-0.3316+\frac{1}{n-1}$$

$$=0.0878+\frac{5}{n-1}.$$

Thus Lemma 2.11 yields

$$0.3316-\frac{1}{n-1}<\left(0.0878+\frac{5}{n-1}\right)^2+0.25$$

which is false for $n>37$.

Thus there must be at least four edges of the pentagon $\mathscr{P}$ on the boundary of $\mathscr{C}$. Hence there can be points of S exterior to $\mathscr{P}$ in at most one of the triangles $\triangle BCD$ or $\triangle ACE$. Thus

$$|\mathscr{C}|-|\mathscr{P}|\leq\tfrac{1}{4}T\leq\tfrac{1}{8}(n-1)t.$$

Hence the number of points of S exterior to $\mathscr{P}$ is no greater than $\frac{1}{8}(n-1)$ and hence (2.8) becomes

$$k_1\geq k-\tfrac{1}{8}(n-1)>0.6658(n-1). \tag{2.8}$$

If there are only four edges of $\mathscr{P}$ on the boundary of $\mathscr{C}$ then there must be an adjacent pair containing more than $0.3329(n-1)$ points of S in contradiction to Lemma 2.11.

Finally, if $\mathscr{C}=\mathscr{P}$, then $k_1=k>0.7908(n-1)$. According to [1] we have

$$|\mathscr{P}|\leq\sqrt{5}T<2.236\ T.$$

Thus Lemma 2.9 can be improved to show that, if any side $\mathscr{E}$ of $\mathscr{P}$ contains $c(n-1)+1$ points of S, then

$$c\leq\tfrac{1}{2}\sqrt{5}-1+\frac{2}{n-1}<0.1180+\frac{2}{n-1}. \tag{2.14}$$

The same argument as in the proof of Lemma 2.2 now yields that

$$k\geq(2-\tfrac{1}{2}\sqrt{5})(n-1)>0.8919(n-1) \tag{2.15}$$

$$>5(0.1180(n-1)+2)$$

for $n>37$, in contradiction to (2.14).

Thus $\mathscr{C}$ has no more that four sides. Hence $|\mathscr{C}|\leq 2T$ and the first part of Theorem 2.1 follows from Lemma 2.7. To prove the affine equivalence of extremal sets to S_0 for even n, divide the quadrilateral $\mathscr{C}$ along a diagonal. One of the two triangular parts, $\mathscr{T}_0$, must contain at least $[\frac{1}{2}(n+3)]$ points of S. Thus triangulation of $\mathcal{T}_0$ yields at least $[\frac{1}{2}(n-1)]$ triangles. If $f(S)=f(n)$, we must have $|\mathcal{T}|=T$ and all points of $\mathcal{T}_0\cap S$ on the boundary of $\mathcal{T}_0$. Since all points of S on the boundary of $\mathcal{C}$ appear on two opposite edges of $\mathcal{C}$, it follows that all but one of the points of $\mathcal{T}_0\cap S$ are on one side of $\mathcal{T}_0$. Since all triangles in the triangulation must have area $t=T/[\frac{1}{2}(n-1)]$ it follows that there are exactly $[\frac{1}{2}(n+1)]$ equally spaced points on one side $\mathcal{A}$ of $\mathcal{C}$. The argument applies equally to the triangle $\mathcal{T}_0$ with side $\mathcal{A}$ and opposite vertex at the other endpoint of the opposite side. Hence $|\mathcal{T}_0|=|\mathcal{T}_0|=T$ and the opposite side, $\mathcal{B}$, is parallel to $\mathcal{A}$, and contains $[\frac{1}{2}n]$ points of S. If n is even this shows that $\mathcal{C}$ is a parallelogram and that the points of $\mathcal{B}$ are also equally spaced.For odd n we can vary the length of $\mathcal{B}$ and the spacing of the $[n/2]$ points on $\mathcal{B}$ as long as $b\leq a$ and none of the intervals on $\mathcal{B}$ has length less than $2a/(n-1)$ where a, b are the lengths of $\mathcal{A}, \mathcal{B}$.

The condition $n>37$ was used primarily in the proof of Lemma 2.2. With the use of the integral part $[\frac{1}{2}(n-1)]$ instead of $\frac{1}{2}(n-1)$ it is easy to prove the result for smaller even n, but it would prove tedious to analyze all cases with $5<n<38$.

We only comment that for small odd n there are other extremal n-tuples. Thus for $n=7$ the set S consisting of the vertices and center of a regular hexagon also yields $f(S)=3$, and for $n=9$ the square 3×3 lattice S also yields $f(S)=4$.

3. Related problems and conjectures

One can pose the analogous problem in higher dimensions.

3.1. Problem. Let S be a set of n points in E^m not all in one hyperplane and let $f_m(S)$ denote the ratio of the maximal and minimal volumes of nondegenerate simplices with vertices in S. Find $f_m(n)=\inf_S f_m(S)$.

In analogy to the solution for the case $m=2$ it is easy to verify that

(3.2) $\quad f_m(n)\leq[(n-1)/m]$

by taking equally spaced points on parallel lines through the vertices of an $(m-1)$-simplex. It is reasonable to conjecture that equality holds in (4.2) for sufficiently large n.

An apparently different problem seems to lead to the same construction.

3.3. Problem. Let S be a set of n points in E^m not all in one hyperplane. What is the minimal number $g_m(n)$ of distinct volumes of nondegenerate simplices with vertices in S?

The above example shows that $g_m(n)\leq[(n-1)/m]$ and we conjecture that equality holds at least for sufficiently large n.

Theorem 1.2 and Lemmas 2.5 and 2.6 suggest various extensions of Sas' results [2].

3.4. Problem. If the inscribed triangle $\mathcal{T}$ of maximal area has one side on the boundary of the convex domain $\mathscr{C}$ what is $\max_{\mathscr{C}} |\mathscr{C}|/|\mathcal{T}|$?

Sas' theorem is valid for the maximal areas of an n-gon, $n \geq 3$, inscribed in a convex curve $\mathscr{C}$ that is, the n-gon contains a maximal proportion of $|\mathscr{C}|$ if and only if $\mathscr{C}$ is elliptic and the n-gon is affine-regular. This leads to generalizations of Problem 3.4.

3.5. Problem. Let $\mathscr{P} = p_1 p_2 \cdots p_n$ be inscribed in the convex curve $\mathscr{C}$ and let $1 \leq i_1 \leq i_2 < \cdots < i_k \leq n$. If the edges $P_{i_j} P_{i_{j+1}}$, $j = 1, 2, \ldots, k$, are on $\mathscr{C}$ what is $\max_{\mathscr{C}} |\mathscr{C}|/|\mathscr{P}|$?

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