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# DISJOINT CLIQUES AND DISJOINT MAXIMAL INDEPENDENT SETS OF VERTICES IN GRAPHS

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In this paper, we find lower bounds for the maximum and minimum numbers of cliques in maximal sets of pairwise disjoint cliques in a graph. By complementation, these yield lower bounds for the maximum and minimum numbers of independent sets in maximal sets of pairwise disjoint maximal independent sets of vertices in a graph. In the latter context, we show by examples that one of our bounds is best possible.

We use notation and terminology of [1]. Throughout this paper, $G$ is a simple finite graph, and $n$ refers to the number of vertices of $G$. $|S|$ is the number of elements in the set $S$. A set $S$ with property $P$ is *maximal* (with respect to $P$) if no set $S'$ exists with $S$ properly contained in $S'$ such that $S'$ has property $P$. A set $S$ with property $P$ is *maximum* (with respect to $P$) if no set $S'$ exists with $|S|<|S'|$ such that $S'$ has property $P$. If $S$ is a vertex or a set of vertices, $N(S)$ is the set of neighbors of $S$ in $G$.

C. Berge (unpublished; see [1, 2]) and independently C. Payan [3] conjectured that any regular graph has two disjoint maximal independent sets of vertices. While this conjecture has now been shown to be false [4, 6], for graphs which are regular of degree $n-k$, Cockayne and Hedetniemi [2] did verify the conjecture for $1 \leq k \leq 7$ and C. Payan [5] for $k \leq 10$. In this paper we show it is true for $k < -2+2\sqrt{2n}$.

Let $B(G)$ be the maximum cardinality of a set of pairwise disjoint maximal independent sets of vertices in $G$. Cockayne and Hedetniemi first introduced a notation for $B(G)$ in [2]. Let $B^c(G)$ be the maximum cardinality of a set of pairwise disjoint maximal cliques in $G$. Let $b(G)$ be the smallest cardinality of a maximal set of pairwise disjoint maximal independent sets of vertices in $G$, and let $b^c(G)$ be the smallest cardinality of a maximal set of pairwise disjoint maximal cliques in $G$. Clearly $B(G) \geqslant b(G)$, $b(G) = b^c(G^c)$, and $B(G) = B^c(G^c)$. Although, in the tradition of Cockayne and Hedetniemi [2], we are primarily interested in $b(G)$ and $B(G)$, our proofs are more easily described for $b^c(G)$ and $B^c(G)$.

On three occassions in the following proof, we will use the inequality $(c_i - \frac{1}{2}(k+g))^2 \geqslant 0$, for integers $g$ and $c_i$, in the form

$$c_i(k+g-c_i) \leqslant \tfrac{1}{4}(k+g)^2. \tag{A}$$

**Theorem 1.** *If $G$ is a graph with $n$ vertices and maximum degree $k$, then*

$$b^c(G) \geqslant 4n/(k+2)^2.$$

*Further, if $G$ is regular of degree $k$, then*

$$b^c(G) \geqslant 8n/(k+3)^2.$$

**Proof.** Set $b^c(G) = b$. Let $C = \{C_1, C_2, \ldots, C_b\}$ be a smallest maximal set of pairwise disjoint maximal cliques in $G$. Set $c_i = |C_i|$ for each $i$, $Z = \bigcup_{i=1}^b C_i$, and $Y = V(G) - Z$. If any vertex $y$ of $Y$ were joined to no members of $Z$, then any clique containing $y$ would be disjoint from $Z$, which is impossible. Also, since each vertex of $C_i$ is adjacent to at most $k - c_i + 1$ vertices of $Y$,

$$\sum_{i=1}^{b} c_i(k - c_i + 1) \geqslant |Y| = n - \sum_{i=1}^{b} c_1.$$

Thus $\sum_{i=1}^b c_i(k+2-c_i) \geqslant n$, or by (A)

$$\tfrac{1}{4}b(k+2)^2 \geqslant n,$$

whence

$$b^c(G) \geqslant 4n/(k+2)^2.$$

Now suppose $y \in Y$ has exactly one neighbor in $Z$. Let that neighbor be $x$ and suppose $x \in C_i$. If $v \in N(y) \cap Y$, then a maximal clique in $G$ containing the edge $vy$ must meet $Z$, and the only possible such meeting is in the vertex $x$, so $xv$ is in $E(G)$. Since $x$ has a neighbor in $G$ not in $Z$, $c_i \geqslant 2$. Hence

$$d_G(x) \geqslant d_G(y) + c_i - 1 > d_G(y).$$

Therefore, if $G$ is regular, every vertex of $Y$ is adjacent to at least two vertices of $Z$. Proceeding as before,

$$\sum_{i=1}^{b} c_i(k+1-c_i) \geqslant 2\,|Y| = 2\left(n - \sum_{i=1}^{b} c_i\right),$$

whence

$$\tfrac{1}{4}b(k+3)^2 \geqslant 2n, \quad \text{or} \quad b^c(G) \geqslant 8n/(k+3)^2,$$

**Corollary.** *If $G$ has minimum degree $n-k$ and $n$ vertices, then*

$$b(G) \geqslant 4n/(k+1)^2.$$

*Further, if $G$ is regular of degree $n-k$, then*

$$b(G) \geq 8n/(k+2)^2.$$

We shall now prove the first inequality in both the theorem and its corollary are best possible. This will be done by showing that for every $b$ and for every even positive integer $k$, there exist graphs $G$ of $n$ vertices and maximum degree $k$, with a maximal set of cardinality $b^c(G)$ of pairwise disjoint maximal cliques such that

$$b^c(G) = 4n/(k+2)^2.$$

Letting $t = \frac{1}{2}(k+2)$, we form a graph $G'$ from one copy of $K_t$ and $t$ disjoint copies of $K_{t-1}$ by assigning to each vertex of $K_t$ one of the copies of $K_{t-1}$ and then joining each vertex of $K_t$ to all of the vertices of its assigned copy of $K_{t-1}$. The resulting graph has maximum degree $k$. Now the disjoint union of $b$ copies of $G'$ is the desired graph $G$.

The corollary to Theorem 1 has the following consequence relative to the work reported in the third paragraph of this paper.

**Corollary.** *Let $G$ be a graph with $n$ vertices and minimum degree $n-k$. If $k < -1 + 2\sqrt{n}$, then $G$ includes two disjoint maximal independent sets of vertices. Further, if $G$ is regular of degree $n-k$ and if $k < -2 + 2\sqrt{2n}$, then $G$ includes two disjoint maximal independent sets of vertices.*

**Theorem 2.** *Let $G$ be a graph with $n$ vertices and maximum degree $k$. Then $B^c(G) \geq 6n/(k+3)^2$.*

**Proof.** Let $H$ be a graph with $V(H) = V(G)$, $E(H)$ as small as possible with $E(H) \subseteq E(G)$ and $B^c(H) = B^c(G) = b$. Let $\{C_1, C_2, \ldots, C_b\}$ be a maximum set of disjoint maximal cliques in $H$ and let $c_i = |C_i|$ for each $i$. Further, choose the set $\{C_1, \ldots, C_b\}$ such that $\sum_{i=1}^{b} c_i$ is as small as possible. Let $Z = \bigcup_{i=1}^{b} C_i$ and let $Y = V(H) - Z$. Let $Y' = \{u_1, u_2, \ldots, u_s\}$ be the set of vertices in $Y$ such that $|N_H(u_i) \cap Z| = 1$.

First we show $Y'$ is independent in $H$. For each $i \in \{1, 2, \ldots, s\}$, let $x_i$ be the member of $N_H(u_i) \cap Z$. Suppose $u \in Y \cap N_H(u_i)$ and suppose $x_i \notin N_H(u)$. Then a maximal clique containing $uu_i$ is disjoint from $Z$, a contradiction. Thus $x_i$ is adjacent in $H$ to every member of $N_H(u_i) \cap Y$. If $u_1u_2 \in E(H)$, then $x_1 = x_2 = x$ and $x$ is adjacent to every member of $N_H(u_1) \cup N_H(u_2)$. Let $H' = H - u_1u_2$. Since $E(H)$ is as small as possible under the given conditions, $B^c(H') \neq B^c(H)$. Now $C_1, \ldots, C_b$ are maximal cliques in $H'$ as well as in $H$, so $B^c(H') > B^c(H)$. Let $D_1, D_2, \ldots, D_b, D_{b+1}$ be $b+1$ pairwise disjoint maximal cliques in $H'$. Since $H$ does not have $b+1$ pairwise disjoint maximal cliques, there exist $D_i$ and $D_j$ such that $u_1 \in D_i$, $u_2 \in D_j$, and $u_1$ is adjacent in $H$ to every vertex in $D_j$ or $u_2$ is adjacent in $H$ to every vertex in $D_i$. Since $x$ is adjacent in $H'$ to every member of

$N_H(u_1) \cup N_H(u_2)$, $x \in D_i \cap D_j$. But this is a contradiction. Hence $Y'$ is independent in $H$.

Choose $y_1 \in Y'$ and suppose its neighbor in $Z$ is $x$. Let $Y'' = N_H(x) \cap Y'$ and suppose $Y'' = \{y_1, \ldots, y_p\}$. Let $\{v_1, \ldots, v_r\} = N_H(x) \cap (Y - Y'')$. Let $C$ be a maximal clique in $H$ containing $xy_1$. Since $N_H(y_1) \cap Z = \{x\}$, $C \subseteq \{x, y_1, v_1, \ldots, v_r\}$. Suppose $x \in C_i$; then $C \cap C_j = \emptyset$ for all $j \in \{1, 2, \ldots, b\} - \{i\}$. since $\sum_{j=1}^{b} c_j$ is a minimum, $|C| \geq |C_i| = c_i$. Hence $r \geq c_i - 2$.

Further, $d_H(x) \geq r + p + c_i - 1$. since $d_H(x) \leq \Delta(H) \leq k$,

$$p \leq k - r - c_i + 1 \leq k - 2c_i + 3. \tag{1}$$

Let $f = |Y'|$. Then, by (1),

$$f \leq \sum_{j=1}^{b} c_j(k - 2c_j + 3). \tag{2}$$

Let $a$ be the number of edges in $H$ with one end in $Z$ and the other end in $Y$. Since any vertex in $C_j$ is joined to at most $k - (c_j - 1)$ elements of $Y$,

$$a \leq \sum_{j=1}^{b} c_j(k - c_j + 1). \tag{3}$$

Since the edges joining vertices in $Y'$ to $Z$ are counted by $f$, and since every vertex of $Y - Y'$ is joined to at least two vertices of $Z$,

$$a \geq 2\left(n - \sum_{j=1}^{b} c_j - f\right) + f. \tag{4}$$

Combining (3) and (4) and applying (2),

$$\sum_{j=1}^{b} (2c_jk - 3c_j^2 + 6c_j) \geq 2n.$$

Multiplying by 3 and applying (A),

$$b(k+3)^2 \geq 6n,$$

or

$$B^c(G) = B^c(H) \geq 6n/(k+3)^2.$$

**Corollary.** *If $G$ is a graph with $n$ vertices and minimum degree $n-k$, then*

$$B(G) \geq 6n/(k+2)^2.$$

**Corollary.** *Every graph with $n$ vertices and minimum degree greater than $n - \sqrt{6n} + 2$ has two disjoint maximal independent sets of vertices.*

Probably the result in the foregoing corollary is not best possible in the sense of having the correct power of $n$ subtracted from $n$; the highest minimum degree we have yet found in a graph with no two maximal independent sets disjoint is approximately $n-(1+\sqrt{2})n^{2/3}$. This example is constructed in the following manner:

Let $p$ be a positive integer and let

$$n=\binom{p+2}{2}+\tfrac{1}{2}p^2(p+2).$$

Let $S_1,\ldots,S_{p+2}$ be disjoint sets of points of cardinality $\frac{1}{2}p^2$ and let $Z=\{z_{ij}: i\neq j$ and $i,j\in\{1,2,\ldots,p+2\}\}$. Then $|Z|=\binom{p+2}{2}$. Form graph $G$ such that $V(G)=Z\cup\bigcup_{i=1}^{p+2}S_i$ and $xy\in E(G)$ iff either $x\in S_i$ and $y\in S_j$ with $i\neq j$ or $x=z_{ij}$ and $y\in S_r$ with $r\notin\{i,j\}$. The maximal independent sets are $Z$ and sets of the form $S_r\cup\{z_{ij}: i=r \text{ or } j=r\}$. It is easy to see no two of these have a non-empty intersection. Furthermore, the minimum degree $\delta$ is the degree of an element of $Z$, so $\delta=\frac{1}{2}p^3$. Since $n^{2/3}\approx(p+1)^2(2^{-2/3})$ and $n-\delta\approx\frac{3}{2}(p+1)^2\approx\frac{3}{2}(2^{2/3})n^{2/3}$, so $\delta\approx n-\frac{3}{2}(2n)^{2/3}$.

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