Solved and unsolved problems in combinatorics and combinatorial number theory

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In this report I discuss as usual some combinatorial problems which interested me and where some progress has been made. I do not claim that I give a survey of the subject but restrict myself to my problems or to problems which interested me during my very long life.

1. The first problem which was (unfortunately) not mine is due to van der Waerden. Van der Waerden conjectured more than 50 years ago that the permanent of a doubly stochastic $m \times m$ matrix is at least $m!/m^m$ - and equally holds if and only if all the entries are $1/m$. For a long time this problem did not get the attention it deserved but for the last 25 years many mathematicians worked on it. Last year Legonihev finally settled the conjecture affirmatively. He used a geometric inequality of Alexandroff-Fenchel on mixed volumes.

2. Nearly 50 years ago Sidon asked me the following question: An infinite sequence of integers $1 \le a_1 < a_2 < \ldots$ is said to have property B_2 if the sums $a_i + a_j$ are all distinct. Sidon observed that there is a B_2 sequence satisfying $a_k < ck^4$ for all k and he asked me to try to improve this. I observed that the greedy algorithm gives the existence of a B_2 sequence satisfying $a_k < ck^3$ and I proved that for every B_2 sequence $\limsup a_k/k^2 = \infty$ and Turán and I constructed a B_2 sequence with $\liminf a_k/k^2 < \infty$.

I could never prove that there is a B_2 sequence for which

$$(1) \qquad \lim a_\ell/k^3 = 0.$$

In fact I conjectured that for every $\varepsilon > 0$ there is a B_2 sequence for which for $k > k_0(\varepsilon)$

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(2) $$a_k < k^{2+\varepsilon} .$$

Rényi and I proved by probabilistic methods that there is a sequence satisfying (2) for which the number of solutions of $a_i+a_j=m$ is less than C_ε.

Last year Ajtai, Komlós and Szemerédi proved (1). More precisely they proved that there is a B_2 sequence for which $a_k < c_1 k^3/\log k$. Their basic tool is a remarkable new result in graph theory which I will state in the next chapter and which will no doubt have many applications.

M. Ajtai, J. Komlós and E. Szemerédi, A dense infinite Sidon sequence, European Journal of Combinatorics 2(1981), 1-11. For the older results on Sidon sequences see the book by H. Halberstam and K. F. Roth, Sequences, Oxford University Press, 1966.

3. Let $G(n;e)$ be a graph of m vertices and e edges. Put $t = \frac{2e}{n}$. Turán's well known theorem easily implies that $\alpha(G) \geq n/t+1$ where $\alpha(G)$ is the largest independent set of our $G(n;e)$. This estimation as it is well known is best possible, to see this let $G(n;e)$ be the union of complete graphs of size $[n/t+1]$. Szemerédi had the lucky and ingenious idea that if we assume that $G(n;e)$ has no triangle then $\alpha(G) > \frac{n}{t+1}$ can perhaps be significantly improved. Ajtai, Komlós and Szemerédi in fact proved the following Theorem: Let $G(n;e)$ be a graph which has no triangle. Then

(1) $$\alpha(G) > \frac{n}{100t} \log t.$$

(1) is best possible apart from the value of the contant $\frac{1}{100}$. (1) was the main tool in constructing dense Sidon sequences. But (1) and its extensions have many further applications. Here is a sample. Let $r(3,n)$ be the smallest integer for which if we color the edges of $K(r(3,n))$ (the complete graph of $r(3,n))$ vertices) by two colors than either there is a monochromatic triangle in color I or a $K(n)$ in color II. It was known that (the lower bound is due to me the upper bound to Graver and Yackel).

(2) $$c_1 \frac{n^2}{(\log n)^2} < r(3,n) < c_2 n^2 \operatorname{loglog} n/\log n$$

Ajtai, Komlós and Szemerédi proved that

$$(3) \qquad r(3,n) < c_3 n^2/\log n$$

At first sight (3) seems only a modest improvement, but there is some hope that perhaps (3) gives the correct order of magnitude of n though the proof of this is nowhere in sight at present.

Komlós, Pintz and Szemerédi using related results for hypergraphs disproved a well known conjecture of Heilbronn. They proved the existence of a set of n points $X_1, \dots, X_n$ in the unit circle so that all the triangles (X_i, X_j, X_ℓ) have area $> c \log n/n^2$.

M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey Numbers, J.C.T. A 29(1981), 354-360.

J. Komlós, J. Pintz and E. Szemerédi. A lower bound for Heilbronn's problem, will appear in J. London Math. Soc.

See also a forthcoming paper of M. Ajtai, J. Komlós, J. Pintz, J. Spencer and E. Szemerédi, Extremal uncrowded hypergraphs.

4. Ajtai, Komlós, Szemerédi and I investigated the following general problem: Let H be a graph. Denote by $f(n;t,H)$ the largest integer for which every $G(n;e), t = 2e/n$ which does not contain H satisfies

$$\alpha(G(n;e)) \geq f(n;t,H).$$

As stated in paragraph 3 Ajtai, Komlós and Szemerédi proved that $f(n;t,K(3)) > \frac{n}{100t} \log t$.

We proved the following theorem:

$$(1) \quad f(n;t,K(p)) > c\,\frac{n}{t} \log A, \text{ where } A = \frac{\log t}{p}\ .$$

(1) shows that for $p = o(\log t)$ the trivial bound ${}^{cn}/t$ for $\alpha(G)$ can be improved by a factor tending to infinity. There are three big gaps in our knowledge in connection with these problems.

The first gap is that $p=o(\log t)$ can perhaps be replaced by $p = o(t^\varepsilon)$ and $\alpha(G)t/n$ will nevertheless tend to infinity. The second big gap is that in (1) log A can perhaps be replaced by log t – at least for every fixed p as $t \to \infty$. For p = 3 this is the result of Ajtai, Komlós and Szemerédi. Unfortunately we can not even settle it for p = 4.

The third big gap concerns hypergraphs. Let $G^{(r)}(n,c)$ be an r-graph with c edges (i.e. n-tuples). Put $t = ({}^{rc}/n)^{1/r-1}$. Spencer proved that

$$(2) \qquad \alpha(G^{(r)}(n;c)) > {}^{cn}/t \quad .$$

Denote by $f(n;t,H^{(r)})$ the largest integer for which every $G^{(r)}(n;c)$ which contains no $H^{(r)}$ has an independent set of size $f(n;t,H^{(r)})$. Ajtai, Komlós, Pintz, Spencer and Szemerédi improved (2) by a factor $(\log t)^{1/r-1}$ if $G^{(r)}(n;c)$ contains no cycle of length ≤ 4. Is it true that

$$f(n;t,K^{(3)}(4))\ \frac{t}{n} \to \infty \ ?$$

This is the third big gap in our knowledge in this fascinating subject.

M. Ajtai, P. Erdős, J. Komlós and E. Szemerédi, On Turán's theorem for sparse graphs, will appear in the new Hungarian Journal Combinatorica.

5. Let G(n;c) be a graph of n vertices and c edges. Let $F_c(n)$ be the number of ways one can color G(n;c) by two colors so that there should be no monochromatic triangle. Clearly $F_c(n)$ is 0 if c is large enough. Rothschild and I conjectured that for $n > n_0$

$$(1) \qquad \max_{G(n)} F_c(n) = 2^{[n^2/4]} \ .$$

The maximum is reached if and only if G(n;c) is Turán's graph. Recently Kostochka proved our conjecture. Many generalizations of (1) are likely to be true, but as far as I know these have not yet been investigated.

Kleitman, Rothschild and I proved that almost all graphs which do not contain a triangle are bipartite. Our results on graphs not containing a K(r) are not in such a final form.

Denote by $f_3(n)$ the number of graphs of n vertices which do not contain a triangle but if we add any new edge to our graph it will contain a triangle. Rothschild and I conjectured several years ago that

$$(2) \qquad \log f_3(n) = (1+o(1)) \frac{n^2}{8} \log 2.$$

The proof of (2) presented unexpected difficulties and is not yet complete. We have no satisfactory upper estimation for $f_3(n)$.

The following related problem seems to be of interest: Rothschild and I proved without much difficulty that if $G(2n)$ is a graph without triangles then the number of maximal independent sets of the vertices of G is at most $2^{n/2}$, equality when $G(n)$ consists of n independent edges.

Let now $G(10n)$ be a graph without triangles each vertex of which has valency ≥ 3. Is it true that the number of maximal independent subsets is at most 15^n? Equality occurs if $G(10n)$ consists of n vertex independent Petersen graphs.

P. Erdős, D. Kleitman and B. Rothschild, Asymptotic enumeration of K_n-free graphs, Coll, Inter. Teoria. Comb. Roma 1973 Atti Con. Linhei No 17 Tomo II 19-27.

6. Faudree, Rousseau, Schelp and I define the size Ramsay number $\hat{r}(G)$ of G as the smallest integer for which there is a graph H having $\hat{r}(G)$ edges, (the number of vertices of H is irrelevant) so that for any coloring of the edges of H by two colors at least one of the colors contains a copy of G. We could not decide whether $\hat{r}(P_n)$ or $\hat{r}(C_n)$ is of linear growth, where P_n is a path of n vertices and C_n a circuit of n vertices).

J. Beck just proved that both $\hat{r}(P_n)$ and $\hat{r}(G_n)$ are of linear growth. The proof of the first inequality $\hat{r}(P_n) < c_1 n$ is surprisingly simple, the proof of $\hat{r}(C_n) < C_2 n$ is much more complicated. Let $T_n(d)$ be a tree of n vertices and maximal valency d. Beck also proved

$$(1) \qquad \hat{r}(T_n^{(d)}) < c_d\, n(\log n)^2.$$

It is possible that the factor $(\log n)^2$ can be omitted in (1).

It would be very interesting to try to characterize the graphs of n vertices $G(n)$ for which $\hat{r}(G_n) < Cn$. Is there a connected graph G of $n(1+\varepsilon)$ edges for which $\hat{r}(G(n)) < C_1 n$? The answer is probably no.

P. Erdős, A. Faudree, C. Rousseau and R. Schelp, The size Ramsay number, Periodica Math. 9(1978), 145-161.

7. V.T. Sós and I started a few years ago to investigate the following problem: Let H be a graph, $f(n; H,\ell)$ be the smallest integer for which every $G(n; f(n;H,\ell))$ either contains H as a subgraph or $\alpha(G(n;f(n;H,\ell))\geq\ell$. Usually we just assumed $\ell=o(n)$ and one of our main problems then can be stated as follows: For which graphs H is $f(n;H,o(n)) = o(n^2)$. Perhaps the most interesting unsolved problem states: Is it true that $f(n;K(2,2,2), o(n)) = o(n^2)$? In other words is it true that for every $c > 0$ and $n > n_0(c)$ every $G(n; c\, n^2)$ with $\alpha(G(n; c\, n^2)) = o(n)$ contains a $K(2,2,2)$? (i.e. a complete tripartite graph with two vertices of each color.) We proved that $K(2,2,2)$ can not be replaced by $K(3,3,3)$. In fact we showed

$$(1) \qquad f(n;K(3,3,3), o(n)) = (1+o(1))\,\frac{n^2}{4}\ .$$

Bollobás, Szemerédi and I proved that

$$(2) \qquad f(n;K(4), o(n)) = (1+o(1))\,\frac{n^2}{8}\ .$$

$f(n;K(3),\ell) = o(n)$ is trivial, since $f(n;K(3),\ell) \leq \frac{n\ell}{2}$.

In a forthcoming paper Hajnal, Szemeredi V.T. Sós and I prove (among others) that $(r\geq 2)$

$$(3) \qquad f(n;K(2r),o(n)) = \frac{m^2}{2}\,\frac{3r-5}{3r-2}\,(1+o(1);\ f(n;K(2r-1),o(n))=\frac{n^2}{2}\,\frac{r-2}{r-1}(1+\ (1))$$

Several unsolved problems remain e.g. is it true that

$$(4) \qquad f(n;K(4), o(n)) \geq \frac{n^2}{8}\ \ ?$$

In other words there is a graph of n vertices and $\frac{n^2}{8}$ edges which has no $K(4)$ and the largest independent set of which is $o(n)$?

Our proof of (3) easily gives that to every $\varepsilon>0$ there is an $\eta>0$ so that

$$(5) \quad f(n;K(4),\eta n) < \frac{n^2}{8}(1+\varepsilon),$$

but so far we could not show that to every $\eta_1>0$ there is an $\varepsilon_1>0$ so that

$$(6) \quad f(n;K(4),\eta_1 n) > \frac{m^2}{8}(1+\varepsilon_1).$$

Perhaps every $G(n;c\, n^2)$ with $\alpha(G) = o(n)$ and which has no $K(4)$ contains a $K(r,r,r)$ for every $n > n_0(r)$.

Many interesting problems remain on hypergraphs. If $G(n)$ is an ordinary graph such that for every fixed $c > 0$ every set of $c\, n$ vertices spans a subgraph of $f(c)(cn)^2$ edges ($f(c)>0$ for every $c>0$) then it is easy to see that if $n > n_0(\ell)$ our $G(n)$ contains a $K(\ell)$. The proof, by induction on ℓ, is almost immediate. It is easy to see that nothing like this holds for hypergraphs. To see this consider the following hypergraph $f(r=3)$: Put $3^m=n$, the vertices of our $G^{(3)}(n)$ are these integers $\sum_{i=0}^{n-1}\varepsilon_i 3^i$, $\varepsilon_i=0,1$ or 2 and the edges are the triples (X,Y,Z) where for some i, ε_i is 0 for X, 1 for Y and 2 for Z. It is easy to see that this $G^{(3)}(n;[\frac{n^3}{24}])$ contains no $K^{(3)}(4)$, in fact it does not even contain a $G^{(3)}(4,3)$, but every subgraph spanned by cn vertices has at least $f(c)(cn)^3$ edges, where $f(c)>0$ for every $c>0$.

Nevertheless an interesting problem remains. Observe that $f(c)\to 0$ as $c\to 0$. Is it true that for every $\alpha>0$ there is a $c_\ell(\alpha)>0$ so that if $G^{(3)}(n)$ is such that every subgraph spanned by $c_\ell(\alpha)n$ vertices contains at least $\alpha(c_\ell(\alpha)n)^3$ edges, then our $G^{(3)}(n)$ contains a $K^{(3)}(\ell)$? I have not even proved that it must contain a $G^{(3)}(4;3)$.

The following further problems can be raised:
Let $G^{(3)}(n)$ be a hypergraph with vertices $X_1,\dots,X_n$. Assume that for every pair (X_i,X_j) there are $c\, n$ elements X_ℓ so that (X_i,X_j,X_ℓ) is an edge of our $G^{(3)}(n)$. Does it then follow that our $G^{(3)}(n)$ contains a

$G^{(3)}(4;3)$? Turån's graph shows that it does not have to contain a $K^{(3)}(4)$ for $c \le {}^1/3$, but if $c > {}^1/3$ then perhaps it will have to contain a $K^{(3)}(4)$. Assume in addition that every set of $c_1 n$ vertices spans $f(c_1)n^3$ triples. Perhaps this further assumption will force a $K^{(3)}(4)$ for every $c > 0$, but I can not even prove that it contains a $G^{(3)}(4;3)$.

P. Erdős and V.T. Sós, Some remarks on Ramsey's and Turån's theorem I and II, Coll. Math. Soc. J. Bolyai 1969.

Combinatorial Theory and its applications, Vol. 2, 395-404 and Studio Math. Acad. Sci. Hungar. 13 (1978).

P. Erdős and V.T. Sós, Problems and results on Ramsey-Turån type theorems, Proc. West Coast Conf. Combinatorics, Graph Theory and Computing, Humboldt State Univ. Arcata Calif. Sept 1979, 17-23.

E. Szemerédi, Graphs without complete quadrilaterals, (in Hungarian), Mat. Lapok 23 (1978), 113-116.

B. Bollobås and P. Erdős, On a Ramsey-Turån type problem, J.C.T. (B) 21(1976), 166-168.

8. Harary posed the following problem. Let $G^{(n)}$ be a graph of n edges, $m(n)$ is the smallest integer for which there is a $G^{(n)}$ for which $K(m(n)) \to (G^{(n)}, G^{(n)})$. Determine $m(n)$, or estimate it as well as possible.

Faudree, Rousseau, Schelp and I proved

$$(1) \quad c_1\, n/\log n < m(n) < c_3\, n/\log n \quad .$$

We could not get an asymptotic formula for $m(n)$, perhaps this will not be easy.

Let $m_r(n)$ be the smallest integer for which there is a $G^{(n)}$ satisfying $K(m_r(n)) \to (G^{(r)}, K(r))$. We proved

$$(2) \quad c_1\, n^{3/5} < m_r(n) < c_2\, n^{2/3} \quad .$$

We suspect that in (2) the upper bound is close to the truth. We further conjectured that for connected $G^{(n)}$

$$K(2n) \to (G^{(n)}, K(3)).$$

Let $M_r(G^{(n)})$ be the smallest integer for which $K(M_r(G^{(n)})) \to (G^{(n)}, K(r))$. We conjecture that for $r \geq 4$ $M_r(G^{(n)})$ is maximal if $G^{(n)^r}$ is as nearly complete as possible. This is definitely false for r=3 (see (2) of paragraph 3).

Our paper on these subjects will be published soon.

9. It is easy to see that if we color the edges of K(n) by n-2 colors then the union of two suitable colors always contains a triangle, but this is generally false for coloring with n-1 colors. On the other hand one can color the edges of K(n) by $1+\frac{\log n}{\log 2}$ colors so that the union of two colors never contains a K(5) (the edges of the same color will be bipartite). Denote by $f(n;r,\ell)$ the largest integer for which if we color the edges of K(n) by $f(n;r,\ell)$ colors then there is always a $K(\ell)$ where edges have at most r colors. It is easy to see that

$$f(2n;2,3) = 2n-2, \ f(2n+1;2,3) = 2n .$$

Further

(1) $f(n;2,4) \geq f(n;1,4) > c\log n/\log\log n$

Unfortunately, I have no better bounds for $f(n;2,4)$. As stated previously $f(n;2,5) < \left[\frac{\log n}{\log 2}\right] + 1$. $f(n;3,4) > cn^{\frac{1}{2}}$ is easy.

Elckes, Furėdi and I have certain preliminary results like $f(n;3,4) = o(n)$, $f(n;4,4) > \frac{5n}{6}$. Many further interesting problems can be stated for hypergraphs. Nothing is published on this subject since our results are too incomplete.

10. J. Pach and I very recently proved the following Ramsey type results. Let f(n) be the smallest integer for which if we color the edges of a K(f(n)) by two colors there always is a complete subgraph K(m), $m \geq n$ of our K(f(n)) and a color, say I, so that the valency of each vertex of K(m) in I is greater than $\frac{m}{2}$. We proved

(1) $$\frac{c_1 n \log n}{\log\log n} < f(n) < c_2 n \log n .$$

If in the definition of f(n) we insist that m = n then our upper bound in (1) has to be replaced by $c_2 n^2$. We believe at present that this is not close to the truth but is due to the fact that we did not yet find the "right" proof.

Our results will be presented in detail at the combinatorial conference held in Eger Hungary in July, 1981.

10. Several mathematicians (Deuber, Neretril-Rődl, Hajnal, Posa and I) proved simultaneously that for every $G_1(n)$ and $G_2(n)$ there is a G for which if we color the edges of G by two colors then either color I contains G_1 as an induced subgraph or color II contains G_2 as an induced subgraph. This was conjectured by Hanson. Denote by $R(G_1,G_2)$ the smallest integer m for which there is such a graph G of m vertices and put

$$f(n) = \max_{G_1(n)G_2(n)} R(G_1,G_2) \quad .$$

The determination (or good estimation) of $R(G_1,G_2)$ and of $f(n)$ seems very difficult, probably much more difficult than the estimation of the ordinary Ramsey functions $r(G_1(n), G_2(n))$. Hajnal and I recently proved

(1) $f(n) < \exp\exp n^{1+\varepsilon}$..

Unfortunately we have no good lower bound for $f(n)$. We have not even proved that

(2) $f(n) > r(n) = r(K(n); K(n))$.

No doubt very much more than (2) holds, we expect that $f(n)^{1/n} \to \infty$. Due to the meagerness and incompleteness of our results we did not publish the details of our proof of (1), which in any case can be easily reconstructed from our paper with Pósa.

Simonovits believes $f(n) < C^n$ and he even believes that $f(n) = r(n)$, he gave some fairly convincing heuristic arguments for his conjecture. We hope to decide the "truth" in a finite time.

P. Erdős, A.Hajnal and L. Pósa, Strong embeddings of graphs into colored graphs, Coll. Math. Soc. J. Bolyai 10 Infinite and finite sets, Kenthely (Hungary) 1973, 585-595.

W. Deuber, Generalizations of Ransey's theorem ibid. 323-332.

11. Galvin and I proved that $n > n_o(r)$ and we color the edges of $K(n)$ so that every color has at most $\varepsilon_r n$ edges then there is a $K(r)$

all of whose edges have different colors. Perhaps there also is a Hamiltonian circuit all whose edges have different color, but we only proved this if every color occurs at most $f(n)$ times where $f(n) \to \infty$ slowly.

12. Denote by $f(n;t)$ the largest integer for which if $X_1, \dots X_n$ are n points in the plane at most t of which are on a line then one can always find at least $f(n;t)$ of them no three of which are on a line. It is easy to see that

$$(1) \quad \left(\frac{2n}{t}\right)^{\frac{1}{2}} \le f(n;t) \le \frac{n}{t} \ .$$

The upper bound in (1) is trivial and the lower bound easily follows from the greedy algorithm. Perhaps $f(n;t) > c\,\frac{n}{t}$ (where c is an absolute constant independent of n and t).

Let $g(n;t)$ be the smallest integer for which there is a set of $g(n;t)$ points no three on a line which are maximal with respect to this property, i.e. if we add any of the other $n-g(n;t)$ points there will be three of them on a line. (1) clearly holds for $g(n;t)$ too, but now I think the lower bound is close to the truth.

More generally $f_\ell(n;t)$ is the largest integer so that we can always find $f_\ell(n;t)$ of the points $X_1, \dots X_n$ no ℓ on a line $(f(n;t) = f_3(n;t))$. $f_\ell(n;t)$ can be defined analogously. It is perhaps worthwhile to study these two functions too.

These problems can be posed in a more abstract setting. Let $|S|=n$, $A_i \subset S$, $2 \le |A_i| \le t$ and assume that every pair (x,y) of elements of S is contained in one and only one A_i. Let $f^*(n;t)$ be the largest integer so that there is a subset S_1 of S, $|S_1|=f^*(n;t)$ and for every i $|A_i \cap S_1| \le 2$. It is easy to see that $f^*(n;t) \le f(n;t)$ and that $f^*(n;t)$ also satisfies (1).

13. Several years ago the following problem occurred to me: Let $K(n)$ be a complete graph of n vertices. Two players alternately choose edges of $K(n)$, if an edge has been chosen by a player his opponent can not choose it also. The game ends if all the edges have been used up. Denote by $G_1(n)$ respectively $G_2(n)$ the graph determined by the edges belonging to the first (respectively to the second) player. The

first player wins if the clique number of $G_1(n)$ is larger than the clique number of $G_2(n)$ (in other words if $G_1(n)$ contains a larger complete graph than $G_2(n)$). For n=2 trivially the first player wins. Is it true that if $n > 2$ then the first player can not enforce a win?

Modify now the rules as follows: The first player wins if $(v(x))$ is the valency (or degree) of the vertex x of G.

$$v(G_1) > v(G_2) \quad \text{where} \quad v(G) = \max_{X \varepsilon G} v(X) .$$

The first player wins for n=2 or 3 and loses for n=4. I do not know what happens for $n > 4$.

Modify the rules once again. A vertex X belongs to G_1 if the valency of X in G_1 is larger than its valency in G_2. The first player wins if he has more vertices than the second player. For which n can the first player win?

Denote by $f(n)$ the largest integer for which the first player can get G_1 satisfying

$$v(G_1) \geq \frac{n}{2} + f(n).$$

It is not hard to prove that $f(n) \to \infty$ but I have no good estimation for $f(n)$ from above or below. Recently J. Beck obtained several very interesting results on games played on graphs and hypergraphs.

I state a few recent problems on combinatorial number theory.

14. Let $1 \leq a_1 \ . \ . \ .$ be an infinite sequence of integers $A(X) = \sum_{a_i < X} 1$ Assume that our sequence is a basis of order r, in other words: every sufficiently large integer is the sum of r or fewer a's. Denote by $A_\ell(X)$ the number of integers not exceeding X which are the sum of ℓ or fewer a's. I conjecture that if $A_\ell(X) = o(X)$ then

(1) $$A_\ell(X)/A_{\ell+1}(X) \to 0 \quad .$$

Rursa observed that for $\ell=1$ this follows from the results of Freiman, but the general case is still open.

G. Freiman, Foundation of a structural theory of set addition, Amer. Math. Soc. Translation of Math. Monographs Vol. 37.

15. Is it true that every interval of length n contains $c\, n/\log n$ distinct multiples of the primes $p, \frac{n}{3} < p < \frac{n}{2}$?

Denote by $A_c(n,m)$ the number of distinct integers X, $m < X \le m+n$ which have a divisor d, $c\,n < d \le n$. Put $f_c(n) = \max_m A_c(m,n)$. Determine $f_c(n)$ or at least $\lim f_c(n)/n$. Is there a $c > 0$ for which this limit is 1? And in fact for which $f_c(n) = n$? Rursa has a simple proof that $f_c(n) = n$ if $c \sim \frac{1}{\log n}$.

P. Erdős and Carl Pomerance, Matching the natural numbers up to n with distinct multiples in another interval, Indag. Math. 42(1980), 147-161.

16. Can one give a necessary and sufficient condition for a sequence of integers $m_1 < m_2 < \dots$ to have the following property: Let $a_1 < a_2 < \dots$ be an infinite sequence of integers for which $a_i + a_j = m_\ell$ has no solutions. Then the density of $a_1 < a_2 < \dots$ is less than ½. Lagasias and Odlyzko proved that the squares have this property.

17. Denote by $T(n)$ the number of divisors of n and let $1 = d_1 < d_2 < \dots < d_{T(n)} = n$ be the sequence of consecutive divisors of n. Is it true that there is an absolute constant C so that for infinitely many n

(1) $$\sum_{i=1}^{T(n)-1} (d_{i+1}/d_i)^2 < C$$

I am sure that (1) remains true if the exponent 2 is replaced by $1+\varepsilon$. C has of course to be replaced by C_ε.

An old conjecture of mine which so far resisted all attacks states that for almost all n (i.e. for all n if we neglect a sequence of density o)

$$\min_{i} d_{i+1}/d_i < 1 + \varepsilon$$

for every $\varepsilon > 0$.

It seems likely that (1) is satisfied say for $n=k!$ or $n=2,3,..p_k$, but I have not been able to prove anything.