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Ramsey-Minimal Graphs for Matchings

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ABSTRACT

This paper investigates $\mathcal{R}(G,H)$ in the special case where G is a t-matching and H is a 2-matching. Here $\mathcal{R}(G,H)$ is the set $\{F|F \to (G,H)$ and $F' \not\to (G,H)$ for each proper subgraph F' of $F\}$.

1. Introduction.

Let F, G, and H be (simple) graphs without isolated vertices. Write $F \to (G,H)$ to mean that if each edge of F is colored red or blue, then either the red subgraph of F contains a copy of G or the blue subgraph contains a copy of H. The class $\mathcal{A}(G,H) = \{F|F \to (G,H)\}$ is essential in Ramsey theory and is non-empty by the classical theorem of F.P. Ramsey. Furthermore, the *generalized Ramsey number* is

$$R(G,H) = \min_{F\in\mathcal{A}(G,H)} |V(F)|$$ and the size Ramsey number is

$$r(G,H) = \min_{F\in\mathcal{A}(G,H)} |E(F)| .$$

In this paper we concern ourselves with the edge minimal members of $\mathcal{A}(G,H)$, called (G,H)-*minimal graphs*. Thus we formally define this family as $\mathcal{R}(G,H) = \{F \in \mathcal{A}(G,H)|F' \not\to (G,H)$ for each proper subgraph F' of $F\}$.

The problem of characterizing the family $\mathcal{R}(G,H)$ for a fixed pair of graphs (G,H) is extremely difficult. It is the purpose of this paper to consider such a characterization for what should be the simplest case, when G is a t-matching and H is a 2-matching.

Before we consider the difficulties involved in any general characterization of $\mathcal{R}(G,H)$, we give some general information of what is known. This information will motivate our looking first at $\mathcal{R}(G,H)$ in the special case when G and H are both matchings.

The pair (G,H) is called *Ramsey-finite* or *Ramsey-infinite* depending on the cardinality of $\mathcal{R}(G,H)$. An early general result was given by Nešetřil and Rödl.

Theorem 1 [*9,10*]. The pair (G,H) is Ramsey-infinite if at least one of the following holds:

(i). G and H are both 3-connected.

(ii). $\chi(G)$ and $\chi(H) \geq 3$.

(iii). G and H are both forests, neither of which is a union of stars.

This theorem leaves an obvious gap when G or H has connectivity two or less and part (iii) is not satisfied. Special cases for graphs which fit in this gap have been considered in other papers [1-8]. In particular the case when G is a matching has been completely settled.

Theorem 2 [*2*]. If G is a matching and H is an arbitrary graph, then the pair (G,H) is Ramsey-finite.

2. <u>The Main Results</u>.

This result and several others in [3, 4, 5] suggest the following conjectures.

(1). If the pair (G,H) is Ramsey-finite, then also the pair $(G \cup \ell S_1 , H \cup mS_1)$ is Ramsey-finite, where jS_1 denotes a j-matching.

(2). The pair (G,H) is Ramsey-infinite, unless both G and H are stars with an odd number of edges or at least one of G or H contains a single edge component.

There are examples of graphs G (and or H) which have single edge components, yet the pair (G,H) is Ramsey-infinite. Hence the converse of (2) definitely fails. The complete classification of those pairs (G,H) which are Ramsey-finite remains a major unsolved problem.

From Theorem 2 we know $\mathcal{R}(G,H)$ is finite when either G or H is a matching. Thus the most natural place to begin with the difficult classification of $\mathcal{R}(G,H)$ is when both G and H are matchings. In the remainder of the paper we consider this classification problem for $G = tS_1$ and $H = 2S_1$.

It is clear that $\mathcal{R}(S_1,H) = \{H\}$. However, the determination of even $\mathcal{R}(2S_1,H)$ is non-trivial. Thus as a special case we consider the finite family $\mathcal{R}(tS_1,2S_1)$. Clearly $(t+1)S_1 \in \mathcal{R}(tS_1,2S_1)$. For convenience let $\mathcal{R}'(tS_1,2S_1) = \mathcal{R}(tS_1,2S_1) - \{(t+1)S_1\}$.

Now we note that $F \in \mathcal{R}'(tS_1,2S_1)$ if and only if each of the following holds.

(a). F contains a t-matching which is maximal.

(b). For each vertex v of F, $F - v$ contains a t-matching.

(c). For each $C_3 \leq F$, $F - C_3$ contains a t-matching.

This characterization is easy to verify. It follows from the following observation. When the edges of F are two-colored such that no $2S_1$ appears as a subgraph of the blue graph, then the blue graph must be a triangle or must have all its edges incident to the same vertex.

We use this characterization to get some information about $\mathcal{R}(tS_1,2S_1)$. First let $F_1 \in \mathcal{R}'(t_1S_1,2S_1)$, $F_2 \in \mathcal{R}'(t_2S_1,2S_1)$, and let $F_1 \cdot F_2$ denote the graph formed from F_1 and F_2 by identifying a fixed vertex of F_1 with a fixed vertex of F_2. Since F_1, respectively F_2, satisfies (a), (b), (c) above for $t = t_1$, respectively t_2, it is easy to check that both $F_1 \cup F_2$ and $F_1 \cdot F_2$ satisfy (a), (b), (c) for $t = t_1 + t_2$. Essentially the converse of this result also holds. It is straightforward to show that if F is connected and $F \in \mathcal{R}'(tS_1,2S_1)$, then F contains no bridges. Thus let $F \in \mathcal{R}'(tS_1,2S_1)$ and in addition have connectivity one, i.e., F has a cut vertex and no bridges. Let w be a cut vertex of F belonging to an end block. Define F_1 as any end block of F containing w and define F_2 as $F - (F_1 - w)$. Note F_2 is a union of the remaining blocks of F, other than F_1. Neither F_1 nor F_2 are edges so that both F_1 and F_2 have edges not incident to w. Since F satisfies (b), $F - w$ has a t-matching so that $F_i - w$ contains a t_i-matching, $t_i \geq 1$, $(i = 1,2)$ such that $t = t_1 + t_2$. But this t_i-matching in $F_i - w$ is a maximal matching in F_i $(i = 1,2)$, otherwise F would contain a matching greater than t, contrary to (a). Also since F satisfies (b) and (c), it follows that F_i $(i = 1,2)$ satisfies (b) and (c) when $t = t_i$. It is interesting to note that each $t_i \geq 2$, since neither F_i can be a C_3. We summarize the consequences of this discussion in the following two theorems.

Theorem 3. Let $F_1 \in \mathcal{R}'(t_1S_1,2S_1)$ and $F_2 \in \mathcal{R}'(t_2S_1,2S_1)$. Then $F_1 \cup F_2, F_1 \cdot F_2 \in \mathcal{R}'((t_1+t_2)S_1,2S_1)$.

Theorem 4. Let F have connectivity one, $F \in R(tS_1, 2S_1)$. Then there exists a partition (t_1, t_2) of t such that $F = F_1 \cdot F_2$ with $F_1 \in R'(t_1 S_1, 2S_1)$ and $F_2 \in R'(t_2 S_1, 2S_1)$.

Corollary 5. Let t_0 be a fixed positive integer and let $H = \{F | F$ is 2-connected and $F \in R(t_0 S_1, 2S_1)\}$. Then $R(t_0 S_1, 2S_1) = H \cup L \cup \{(t_0 + 1)S_1\}$ where $L = \{L | L = F_1 \cdot F_2$ or $L = F_1 \cup F_2$ with $F_1 \in R'(t_1 S_1, 2S_1)$, $F_2 \in R'(t_2 S_1, 2S_1)$, and (t_1, t_2) a partition of $t_0\}$.

Since it is clear that for each $\ell \geq 2$, $C_{2\ell+1} \in R(\ell S_1, 2S_1)$, the following corollary is a specialization of Theorem 3.

Corollary 6. Let G be a graph with its blocks $B_1, B_2, \ldots, B_k$ being the odd cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$, with each $i_j \geq 5$, such that $\Sigma_{j=1}^{k} \frac{i_j - 1}{2} = t$. Then $G \in R(tS_1, 2S_1)$.

This last corollary does produce a fairly large subset of graphs in $R(tS_1, 2S_1)$. For example the graphs of $R(8S_1, 2S_1)$ which have four different C_5's as their only blocks are listed in Figure 1. From Corollary 5 it is apparent that $R(tS_1, 2S_1)$ is completely determined by its 2-connected members. Even these could prove very difficult to find; for example, $H_1 \in R(6S, 2S_1)$ and $H_2 \in R(10S_1, 2S_1)$ with H_1 and H_2 shown in Figure 5.

We give a few additional lists of $R(G,H)$ for very special small graphs G and H. The case analysis involved in obtaining the lists is additional evidence of the complexity of the problem under discussion.

Theorem 7. Let G_i denote the collection of graphs listed in figures 3 - 7. Then

(i). $R(2S_1, 2S_1) = \{C_5, 3S_1\}$;

(ii). $R(3S_1, 2S_1) = \{4S_1, C_7\} \cup G_1$;

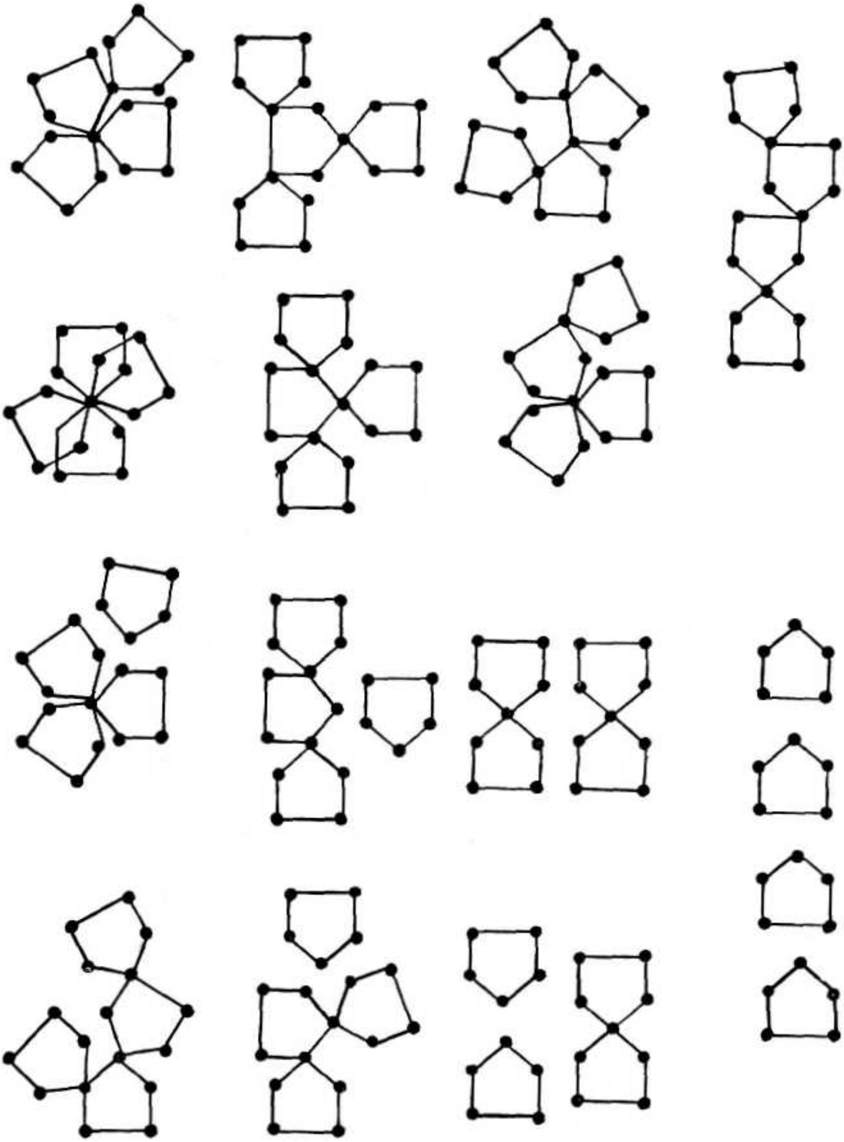
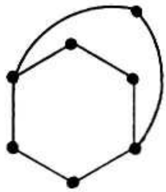
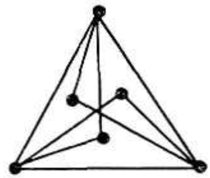


Figure 1.



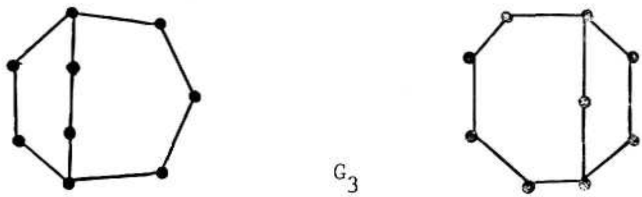
G_1

Figure 2.



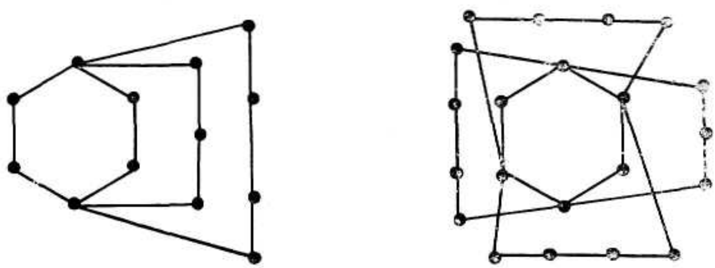
G_2

Figure 3.



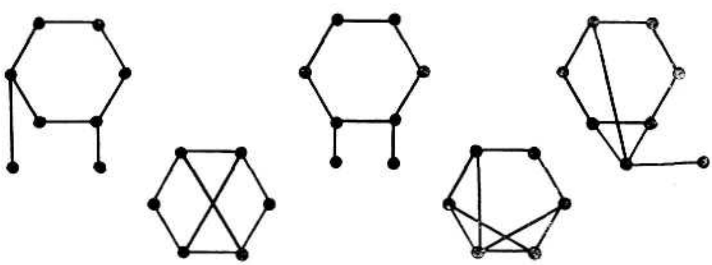
G_3

Figure 4.



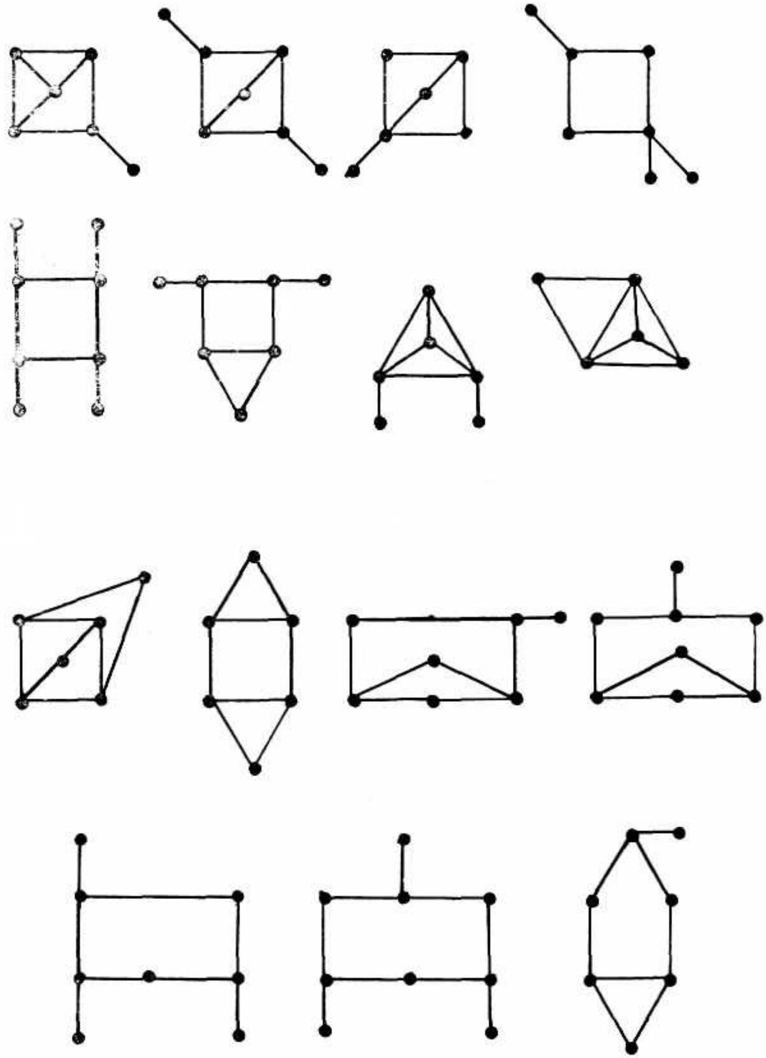
H_1 H_2

Figure 5.



G_4

Figure 6.



G_5

Figure 7.

(iii). $\mathcal{R}(4S_1,2S_1) = \{5S_1, C_5 \cup C_5, C_5 \cdot C_5, C_9\} \cup \mathcal{G}_3$;
(iv). $\mathcal{R}(2S_1,S_2) = \{2S_2, C_4, C_5\}$;
(v). $\mathcal{R}(3S_1,S_2) = \{3S_2, C_4 \cup S_2, C_5 \cup S_2, C_7, C_8\} \cup \mathcal{G}_4$;
(vi). $\mathcal{R}(2S_1,S_3) = \{2S_3\} \cup \mathcal{G}_5$; and
(vii). $\mathcal{R}(2S_1,K_3) = \{K_5, 2K_3\} \cup \mathcal{G}_2$.

There are some obvious questions concerning $\mathcal{R}(tS_1,2S_1)$. For instance, what is the maximum order and size of members of $\mathcal{R}(tS_1,2S_1)$? A rather large upper bound is given for the size in [2].

The results on $\mathcal{R}(tS_1,2S_1)$ given above demonstrate the difficulty in finding an explicit characterization for $\mathcal{R}(G,H)$ for arbitrary G and H . It would be extremely valuable to complete such a characterization for the special pair $(tS_1,2S_1)$.

Another direction of interest would be to find properties common to a fixed family $\mathcal{R}(G,H)$. This might prove fruitful for the more special class $\mathcal{R}(tS_1,2S_1)$.

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