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On the Bandwidths of a Graph and its Complement

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ABSTRACT

The *bandwidth* $b(G)$ of a graph G is defined by

$$b(G) = \min_{\lambda} \max_{e=\{x,y\}} |\lambda(x)-\lambda(y)|$$

where e ranges over all edges of G and λ ranges over all 1 - 1 functions $\lambda:V(G) \to Z^+$, the positive integers. In this note we show for any graph G on n vertices (with $\bar{G}$ denoting its complement),

$$b(G) + b(\bar{G}) \geq n - 2 .$$

Furthermore, for all $n \geq 3$ there exist graphs which achieve this bound.

We also prove:

(i) $b(G) + b(\bar{G}) < 2n - c_1 \log n$, for all graphs G on n vertices;

(ii) $b(G) + b(\bar{G}) > 2n - c_2 \log n$, for almost all graphs G on n vertices.

1. Introduction.

For undefined graph theory terminology see [1] or [8]. The *bandwidth* $b(G)$ of a graph G is defined to be the least integer b such that for some labelling λ of the vertices of G

with distinct integers,

$$|\lambda(x)-\lambda(y)| \leq b \quad \text{for all edges } \{x,y\} \text{ of } G. \tag{1}$$

If λ satisfies (1) and $\lambda(v_1) < \lambda(v_2) < \ldots < \lambda(v_n)$ where n is the number of vertices of G then the labelling $\bar{\lambda}(v_k) = k$, $1 \leq k \leq n$, also satisfies (1). Hence, we need only consider $1-1$ mappings $\lambda: V(G) \to \{1,2,\ldots,n\} \equiv [n]$ for determining $b(G)$.

A number of papers have appeared (e.g., [2], [3], [5], [9], [10], [11]) recently which deal with the bandwidth of a graph, both from the graph theoretic as well as the algorithmic point of view. For example, it has been shown [5] that the problem of determining the bandwidth of a tree is already NP-complete. (For a discussion of this concept, see [6].) For a survey of many of these and related results, the reader can consult [2] or [3].

In this paper we investigate the relationship between $b(G)$ and $b(\bar{G})$ where $\bar{G}$ denotes the complement of G, i.e., $V(\bar{G}) = V(G)$ and $\{x,y\} \in E(\bar{G})$ iff $\{x,y\} \notin E(G)$. It is clear that if G has a small bandwidth then it must have relatively few edges. Consequently $\bar{G}$ has many edges and thus, $b(\bar{G})$ is large. Our purpose is to make this rough notion precise.

2. The Lower Bound.

For a graph G, the *kth* power G^k of G is defined to be the graph which has the same vertex set as G and in which $\{x,y\}$ is an edge iff x and y are connected in G by a path of length at most k. Let P_k denote a path with k vertices. It follows at once from the definition of bandwidth that:

Fact. If G has n vertices then $b(G) \leq b$ iff $G \subseteq P_n^b$.

In particular, it follows that

$$b(P_n^b) = b. \tag{2}$$

Theorem 1. If G has n vertices then

$$b(G) + b(\bar{G}) \geq n - 2. \tag{3}$$

Proof. To simplify the notation we restrict our attention to the case that $n = 2m$. The case in which n is odd follows in exactly the same way. We claim that (3) is an immediate consequence of the following result. (In the remaining part of this paper, we use $\bar{P}_{2m}^{m-1}$ to denote the complement of P_{2m}^{m-1}.)

Lemma.

$$b(\bar{P}_{2m}^{m-1}) = m - 1. \tag{4}$$

Proof. Suppose (4) holds. If $b(G) = b \leq m - 2$ then by the Fact,

$$G \subseteq P_{2m}^{b} \subseteq P_{2m}^{m-1} .$$

Thus,

$$\bar{G} \supseteq \bar{P}_{2m}^{m-1}$$

and by (4),

$$b(\bar{G}) \geq b(\bar{P}_{2m}^{m-1}) = m - 1 .$$

Hence, at least one of G, $\bar{G}$ has bandwidth $\geq m - 1$. Assume

$$b(G) = b \geq m - 1. \tag{5}$$

Therefore,

$$G \subseteq P_{2m}^{b},\ \bar{G} \supseteq \bar{P}_{2m}^{b} . \tag{6}$$

But (5) implies $2m - b - 1 < b + 2$. Since in the case $\bar{P}_{2m}^{b}$ and $\bar{P}_{4m-2b-2}^{2m-b-2}$ are *isomorphic* then

$$b(\bar{G}) \geq b(\bar{P}_{4m-2b-2}^{2m-b-2}) = 2m - b - 2$$

and (3) holds as required.

The remainder of the proof of the theorem will be devoted to proving (4)

To fix notation, let us write the vertex set of $\bar{P}_{2m}^{m-1}$ as $\{X_1,\ldots,X_m,Y_1,\ldots,Y_m\}$ with the edges of $\bar{P}_{2m}^{m-1}$ as all pairs $\{X_i,Y_j\}$, $1 \leq i \leq j \leq m$. The following labelling λ shows that $b(\bar{P}_{2m}^{m-1}) \leq m - 1$:

$$\lambda(Y_i) = i, \qquad 1 \le i \le m-1,$$
$$\lambda(X_i) = i + m - 1, \ 1 \le i \le m-1,$$
$$\lambda(Y_m) = 2m - 1, \qquad \lambda(X_m) = 2m.$$

It remains to show $b(\bar{P}_{2m}^{m-1}) \ge m - 1$.

Suppose the contrary, i.e., assume $b(\bar{P}_{2m}^{m-1}) \le m - 2$. Thus, by the Fact,

$$\bar{P}_{2m}^{m-1} \subseteq P_{2m}^{m-2}$$

i.e.,

$$P_{2m}^{m-1} \supseteq \bar{P}_{2m}^{m-2} .$$

Let $\mu: \bar{P}_{2m}^{m-2} \to P_{2m}^{m-1}$ be an embedding of $\bar{P}_{2m}^{m-2}$ into $\bar{P}_{2m}^{m-1}$. Note that P_{2m}^{m-1} can be formed by starting with a copy of $\bar{P}_{2m-2}^{m-2}$ on the vertex set $\{A_1,\ldots,A_{m-1},B_1,\ldots,B_{m-1}\} = A \cup B$, forming complete graphs on A and B, and adjoining two additional points A* and B*, with A* joined to all points of A and B* joined to all points of B. For ease of notation, let us use [2m] for the vertex set of $\bar{P}_{2m}^{m-2}$. For convenient future reference, we show $\bar{P}_{2m}^{m-2}$ and P_{2m}^{m-1} in Figure 1. Let X denote $\{1,2,\ldots,m\}$ and let Y denote $\{m+1,\ldots,2m\}$.

To begin with, suppose there exist $i,j \in X$ such that $\mu(i) = A^*$, $\mu(j) = B^*$. In this case, however, in $\bar{P}_{2m}^{m-2}$ the vertex 2m is adjacent to every $x \in X$ (which we will occasionally write as $2m \sim x$). Since no vertex in P_{2m}^{m-1} is adjacent to both A* and B* then we have a contradiction.

In the same way, it is impossible that for $i,j \in Y$, $\mu(i) = A^*$ and $\mu(j) = B^*$.

Next, suppose $\mu(m) = A^*$, $\mu(j) = B^*$ for some $j \in Y$. Since $m \sim 1$ in $\bar{P}_{2m}^{m-2}$ then $\mu(1) = A_i$ for some i. However, this is impossible since $1 \sim j$ in $\bar{P}_{2m}^{m-2}$ and consequently

$$\mu(1) = A_i \sim \mu(j) = B^*.$$

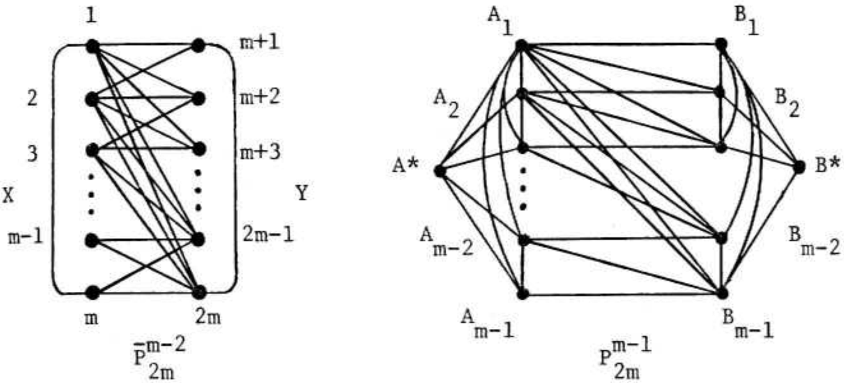

Figure 1

Similarly, we cannot have $\mu(2m) = B^*$, $\mu(i) = A^*$ for some $i \in X$.

Thus, by the symmetry of $\bar{P}_{2m}^{m-2}$ (under $i \leftrightarrow 2m+1-i$) we can assume:

$$\mu(i) = A^* \text{ for some } i \in \{2,3...,m\},$$
$$\mu(j) = B^* \text{ for some } j \in \{m+1,...,2m-1\}.$$

The neighbors of i in $\bar{P}_{2m}^{m-2}$ must be mapped into A; these are $\{m+i-1, m+i, ..., 2m\} \equiv Y'$. Similarly the neighbors of j must be mapped into B; these are $\{1,2,...,j-m+1\} \equiv X'$.

It is important to note that since $\mu(i) = A^*$ is not adjacent to $\mu(j) = B^*$ in P_{2m}^{m-1} then we cannot have $i \sim j$ in $\bar{P}_{2m}^{m-2}$. Thus,

$$j - i \leq m - 2$$

and so, the subgraph in $\bar{P}_{2m}^{m-2}$ induced by X' and Y' is a complete bipartite subgraph (i.e., $x \in X'$, $y \in Y'$ implies $x \sim y$). Informally, the situation is shown in Figure 2.

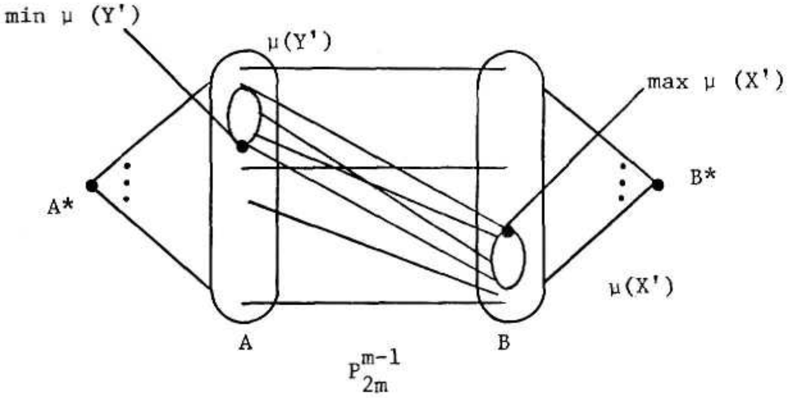

Figure 2.

In fact, we have little more than this. Note that

$$|Y'| = |\mu(Y')| = m - i + 2,$$

$$|X'| = |\mu(X')| = j - m + 1.$$

Since A and B span a copy of $\bar{P}_{2m-2}^{m-2}$ then in Figure 2 $\min \mu(Y')$ (the element A_k of $\mu(Y')$ having the largest index k) must be at least as high as $\max \mu(X')$. Therefore

$$|\mu(Y')| + |\mu(X')| - 1 \leq m - 1$$

so that

$$j - i \leq m - 3. \tag{7}$$

Define

$$U = \{j-m+2, j-m+3, \ldots, i-1\},$$

$$V = \{j+1, j+2, \ldots, i+m-2\}.$$

Thus,

$$|U| = |V| = i - j + m - 2 \equiv t \geq 1.$$

Further, define partitions of U and V by:

$$U = U_1 \cup U_2, \quad V = V_1 \cup V_2$$

where

$$\mu(U_1) \subseteq A, \ \mu(U_2) \subseteq B,$$

$$\mu(V_1) \subseteq A, \ \mu(V_2) \subseteq B.$$

Note that the graph spanned by U and V in $\bar{P}_{2m}^{m-2}$ is isomorphic to $\bar{P}_{2t}^{t-1}$. Also $U \sim Y'$ and $V \sim X'$ in $\bar{P}_{2m}^{m-2}$ (i.e., $u \in U$, $y' \in Y'$ implies $u \sim y'$, etc.).

There are two cases:

(i). $|U_2| + |V_2| \geq t$.

Consider the *level* α of $\min \mu(Y')$, i.e.,

$$\alpha = \max \{i : \mu(y') = A_i \text{ for some } y' \in Y'\}.$$

Partition $\mu(V_2)$ into two pieces:

$$\mu(V_2) = \mu(V_2') \cup \mu(V_2'')$$

where $\mu(V_2')$ consists of all points in $\mu(V_2)$ with level $\geq \alpha$ and $\mu(V_2'')$ consists of all points in $\mu(V_2)$ with level $< \alpha$. Note that since $U \sim Y'$ then $U_2 \sim Y'$. Hence, $\mu(U_2)$ has level $\leq \alpha$.

Similarly, partition $\mu(U_1)$ into $\mu(U_1')$, those points in $\mu(U_1)$ with level $> \alpha$ and $\mu(U_1'')$, those points with level $< \alpha$ (no point in $\mu(U_1)$ can have level α since $\min \mu(Y')$ does). Summarizing:

$$\text{level } \mu(U_1') > \alpha, \ \text{level } \mu(U_1'') < \alpha$$
$$\text{level } \mu(V_2') \geq \alpha, \ \text{level } \mu(V_2'') < \alpha$$
$$\text{level } \mu(Y') \geq \alpha, \ \text{level } \mu(X') \leq \alpha.$$

Claim. $|U_1'| \geq |V_2'|$.

Suppose not, i.e., suppose $|U_1'| < |V_2'|$. Then

$$|U_1''| = t - |U_1'|$$

so that

$$|U_1''| + |V_2'| > t,$$

i.e.,

$$|\mu(U_1'')| + |\mu(V_2')| > t. \tag{8}$$

But we have already noted that the graph in P_{2m}^{m-1} between $\mu(U)$ and $\mu(V)$ is isomorphic to $\bar{P}_{2t}^{t-1}$. Hence, by (8) some point in $\mu(U_1'')$ must be adjacent to some point in $\mu(V_2')$. However, this is *impossible* since level $\mu(U_1'') < \alpha$ and level $\mu(V_2') \geq \alpha$. This proves the Claim.

Finally, we have in A at least $|\mu(Y')| + |\mu(U_1')|$ points with level $\geq \alpha$. In B there are at least

$$|\mu(X')| + |\mu(U_2)| + |\mu(V_2'')|$$

points with level $\leq \alpha$. Since the total number of points in A (and also in B) is just $m - 1$ and $n \geq y + 3$ then we must have

$$|\mu(Y')| + |\mu(U_1')| + |\mu(X')| + |\mu(U_2)| + |\mu(V_2'')| - 1 \leq m - 1 \tag{9}$$

(the -1 term on the LHS coming from the possibility that both A and B may contribute a point of level α). Substituting for these various cardinalities, we obtain,

$$m - i + 2 + |\mu(U_1')| + j - m + 1 + |\mu(U_2)| + |\mu(V_2'')| \leq m,$$
$$j - i + 3 + |\mu(V_2')| + |\mu(V_2'')| + |\mu(U_2)| \leq m \quad \text{(by Claim)},$$
$$j - i + 3 + |\mu(V_2)| + |\mu(U_2)| \leq m \quad (\text{since } V_2 = V_2' \cup V_2''),$$
$$j - i + 3 + t \leq m \quad \text{(by the Case (i) assumption)},$$
$$j - i + 3 + (i-j+m-2) \leq m \quad \text{(by the definitions of } t),$$

i.e.,

$$i \leq 0$$

which is a contradiction. This completes the analysis of Case (i).

(ii). $|U_2| + |V_2| < t$.

The arguments for this case are quite parallel to those for Case (i) and will not be given. As mentioned earlier, when m is odd the arguments are essentially the same (in fact, slightly easier). This completes the proof of Theorem 1. ∎

Corollary.

$$b(\bar{P}_n^r) = n - r - 2 \quad \text{for} \quad r > 0.$$

Proof. Since $b(P_n^r) = r$ then by (3)

$$b(\bar{P}_n^r) \geq n - r - 2. \qquad ∎$$

The labelling which achieves this bound is not difficult to construct and is left to the reader.

Remark. We point out tnat E.C. Milner and N. Sauer [10] and J. Kahn and D.J. Kleitman [9] have recently independently also proved Theorem 1.

3. Upper Bounds.

Since any graph G on n points has bandwidth less than n then it is immediate that

$$b(G) + b(\bar{G}) < 2n.$$

The next two results improve this estimate considerably.

Theorem 2. There is a $c_1 > 0$ such that for all n, every graph G on n vertices satisfies

$$b(G) + b(\bar{G}) \leq 2n - c_1 \log n \tag{10}$$

Proof. A basic result in Ramsey theory (see [4] or [7]) asserts that any 2-coloring of the edges of K_n, the complete graph on n vertices, contains a monochromatic K_z with $z \geq \frac{\log n}{\log 4}$.

Since, the decomposition of K_n into G and $\bar{G}$ can be regarded as a 2-coloring of the edges of G, then either G or $\bar{G}$ contains a K_z. Assume without loss of generality it is G. Thus, $\bar{G}$ contains z points $\{x_1, x_2, \ldots x_z\}$ which span no edge. Consequently $\bar{G}$ has bandwidth at most $n - [\frac{z}{2}]$ (use the highest and lowest $\frac{z}{2}$ labels on the x_k) and so

$$b(G) + b(\bar{G}) < 2n - c \log n$$

for an appropriate $c > 0$. This proves the theorem. ∎

The next result shows that up to the choice of c, (10) is best possible.

Theorem 3. There is a $c_2 > 0$ such that for every n there exists a graph G on n vertices such that

$$b(G) + b(\bar{G}) \geq 2n - c_2 \log n.$$

Proof. It is well known (e.g., see [4] or [7]) that the edges of the complete graph K_n can be 2-colored so that the largest monochromatic complete bipartite subgraph $K_{x,x}$ has $x < c_1 \log n$ for some absolute constant $c_1 > 0$. Define G to be the subgraph consisting of the edges of one of the colors (so that $\bar{G}$ is made up of the edges of the other color). Thus

$$y \geq c_1 \log n \Rightarrow K_{y,y} \not\subseteq G, \bar{G}\ .$$

However, $K_{y,y} \not\subseteq G$ implies $b(\bar{G}) \geq n - 2y + 1$. (Just consider the vertices with labels $1, 2, \ldots, y$ and $n, n-1, \ldots, n-y+1$; some edge spanned by a vertex in each class must be in $\bar{G}$.) Taking $c' = 2c_1$, the theorem is proved. ∎

With a more careful analysis, it is possible to improve the values of the constants in (10) and (11). The exact value would seem to depend on knowing the asymptotic behavior of Ramsey numbers, a problem well known to present difficulties.

We conclude with the observation that if K_n is decomposed in an arbitrary number of edge-disjoint subgraphs

$$K_n = G_1 \cup G_2 \cup \cdots \cup G_k$$

then

$$\sum_{i=1}^{k} b(G_i) \geq \frac{1}{2}n + o(1)n.$$

Furthermore it is easy to see (by decomposing K_n into paths) that this bound can be achieved.

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