MATHEMATICAL NOTES

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FAT, SYMMETRIC, IRRATIONAL CANTOR SETS

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A familiar class of symmetric Cantor sets is obtained as follows: Let $\alpha \in (0,1]$; from $[0,1]$ remove a segment of length $\alpha/3$ to leave two intervals of equal length; from each of these intervals remove a segment of length $\alpha/3^2$ to leave 2^2 intervals of equal length; iterate this process and denote the Cantor set that remains by C_α. This class of Cantor sets is a very fruitful source of examples. An introduction to these Cantor sets and their corresponding Cantor functions can be found in [**1**]; however, this note does not depend on [**1**]. This note shows that, except for a first category set of α's in $[0,1]$, $(0,1) \cap C_\alpha$ contains only irrational numbers. Actually, we show that if $x \in (0,1)$ then the set $[x]$ of α's in $[0,1]$ for which $x \in C_\alpha$ is a closed, nowhere dense subset of $[0,1]$; consequently, $\cup_{x \in A}[x]$ is a first category subset of $[0,1]$ whenever A is a countable subset of $(0,1)$. Letting A be the set of rationals in $(0,1)$ produces irrational Cantor sets.

Focus on the construction of C_α and observe that a point $x \in C_\alpha$ is the intersection of a nested sequence of intervals. Thus x is determined (uniquely) by specifying whether a left or a right subinterval contains x at each step: for $0 < \alpha \leqslant 1$ there is a one-to-one correspondence between the elements $x \in C_\alpha$ and the elements $S \in \mathbb{S}$, where $\mathbb{S}$ denotes the set of subsets of the set N of positive integers; $x \in C_\alpha$ corresponds to the set S_x of positive integers n such that x is in a right subinterval at step n. For $0 < \alpha \leqslant 1$, let ϕ_α denote the map that takes $S_x \in \mathbb{S}$ to $x \in C_\alpha$. For future reference, notice that if x_n denotes the left endpoint of the nth step interval that contains x then $x_1 \leqslant x_2 \leqslant \cdots \to x$. For $\alpha = 0$, there may be two subsets of N corresponding to $x \in C_0 = [0,1]$; for example, $\frac{1}{2}$ corresponds to both the one-element set $\{1\}$ and its complement. Nevertheless, we can define $\phi_0 : \mathbb{S} \to C_0$ as we did for $0 < \alpha \leqslant 1$.

For $S \subset N$, let $\lambda(S) = 2\Sigma_{n \in S} 3^{-n}$ and $\mu(S) = \Sigma_{n \in S} 2^{-n}$; in particular, $\lambda(\phi) = \mu(\phi) = 0$. Then $\lambda(\mathbb{S}) = C_1$ and $\mu(\mathbb{S}) = C_0$; these are the extreme cases $\alpha = 1$ and $\alpha = 0$. Notice that $(\frac{2}{3} - \frac{1}{2}) = \Sigma_{1 < n < \infty}(2^{-n} - 2 \cdot 3^{-n})$; so $\lambda(S) < \mu(S)$ if $1 \notin S \neq \varnothing$ and $\mu(S) < \lambda(S)$ if $1 \in S \neq N$.

Continue to focus on the construction of C_α. After step one, two intervals of length $l_1 = 2^{-1}(1 - \alpha/3)$ remain; after step two, four intervals of length $l_2 = 2^{-1}(l_1 - \alpha/3^2)$ remain. Continuing, one sees that, after each step n, 2^n intervals of length l_n remain, where

$$\begin{aligned} l_n &= 2^{-1}(l_{n-1} - \alpha/3^n) \\ &= 2^{-n}\left(1 - (\alpha/3)\left[1 + (2/3) + \cdots + (2/3)^{n-1}\right]\right) \\ &= 2^{-n}(1 - \alpha) + 3^{-n}(\alpha). \end{aligned}$$

Next notice that if an integer $n \in S \in \mathbb{S}$ and if $x \in C_\alpha$ corresponds to S then $x_{n+1} - x_n = l_n + \alpha/3^n = 2^{-n}(1 - \alpha) + 2 \cdot 3^{-n}(\alpha)$; so $x = \phi_\alpha(S)$, where $\phi_\alpha = (1 - \alpha)\mu + \alpha\lambda$. Thus the map ϕ_α

takes $\mathbb{S}$ onto C_α, and it possesses nice properties. For instance, $\|\phi_\alpha - \phi_\beta\|_\infty = |\alpha - \beta|/6$; so the set $[x]$ of α's in $[0,1]$ for which $x \in C_\alpha$ is a closed subset of $[0,1]$. (If $d = d(x, C_\alpha) > 0$ then $x \notin C_\beta$ for $|\beta - \alpha| < 6d$.) Another relevant property of ϕ_α is that if $\alpha \neq \beta$ and $\varnothing \neq E \neq N$ then $\phi_\alpha(E) - \phi_\beta(E) = (\beta - \alpha)[\mu(E) - \lambda(E)] \neq 0$.

Define a linear ordering $<$ on $\mathbb{S}$ as follows: $E < F$ if there exists a positive integer n such that $E_{n-1} = F_{n-1}$ and $E_n \subsetneq F_n$, where $H_k = H \cap \{0, 1, \ldots, k\}$, $H \in \mathbb{S}$, $k \geq 0$ (i.e., $E < F \Leftrightarrow \phi_\alpha(E) < \phi_\alpha(F)$, $0 < \alpha \leq 1$).

Now we are ready to show that if $0 < x < 1$ then $[x]$ is a nowhere dense subset of $[0,1]$. Suppose $\alpha, \beta \in (0,1)$, $0 < x < 1$ and $\phi_\alpha(E) = x = \phi_\beta(F)$. Also, without loss of generality, suppose that $E < F$. Let n be the smallest positive integer in $F - E$. Let

$$G = E_n \cup \{n+1, n+2, \ldots\}.$$

Then, for $0 < \gamma \leq 1$, $U_\gamma = (\phi_\gamma(G), \phi_\gamma(F_n))$ is a component of $[0,1] - C_\gamma$. Moreover, $\phi_\beta(G) < \phi_\beta(F_n) \leq x \leq \phi_\alpha(G) < \phi_\alpha(F_n)$. Thus, since U_γ deforms continuously from U_α to U_β as γ moves from α to β, there are γ's between α and β for which $x \notin C_\gamma$ (e.g., $x \notin C_\gamma$ when $0 < \phi_\gamma(F_n) - x < \inf\{\phi_\lambda(F_n) - \phi_\lambda(G);\ \lambda \text{ between } \alpha \text{ and } \beta\}$).

One of the referees of this note suggested using a nice subset K of the unit square to display the setting. To obtain K, draw line segments between points $(\lambda(E), 1)$ and $(\mu(E), 0)$, $E \in \mathbb{S}$, and let K denote the union of these intervals. One sees quickly that K is closed, that C_α is the intersection of K with the horizontal line $y = \alpha$, and that $[a]$ is the intersection of K with the vertical line $x = a$. Because the linear measure of C_α is $1 - \alpha$, some of those irrational Cantor sets are fat.

Reference

1. R. B. Darst, Some Cantor sets and Cantor functions, Math. Mag., 45 (1972) 2–7.

ON THE MONOTONICITY OF A CLASS OF EXPONENTIAL SEQUENCES

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It is well known that the sequence $(1 + 1/n)^n$ increases to e, whereas it is somewhat less familiar that the sequence $(1 + 1/n)^{n+1}$ decreases to e [3]. This note concerns the monotonicity of the sequence

$$a_n = (1 + 1/n)^{n+\alpha} \quad \text{for } 0 < \alpha < 1.$$

To this end, a sequence $\{\beta_k\}$ is defined by

$$\left[\frac{(k+1)^2}{k(k+2)}\right]^{\beta_k} = \left[\frac{k(k+2)}{(k+1)^2}\right]^{k+1}\left(\frac{k+1}{k}\right)$$

for $k = 1, 2, \ldots$. The value of β_k is precisely the value of α required for $a_k = a_{k+1}$. Several properties of $\{\beta_k\}$ will be essential.

LEMMA 1. *The sequence $\{\beta_k\}$ increases.*

Proof. Since

$$\beta_{k-1} = \frac{k \ln((k^2-1)/k^2) + \ln(k/(k-1))}{\ln(k^2/(k^2-1))},$$

we are led to consider the function $y = F(x)$ with