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Completeness Properties of Perturbed Sequences

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If S is an arbitrary sequence of positive integers, let $P(S)$ be the set of all integers which are representable as a sum of distinct terms of S. Call S *complete* if $P(S)$ contains all large integers, and *subcomplete* if $P(S)$ contains an infinite arithmetic progression. It is shown that any sequence can be perturbed in a rather moderate way into a sequence which is not subcomplete. On the other hand, it is shown that if S is any sequence satisfying a mild growth condition, then a surprisingly gentle perturbation suffices to make S complete in a strong sense. Various related questions are also considered.

1. INTRODUCTION

Let S be an arbitrary sequence of positive integers. Define $P(S)$ to be the set of all integers which are representable as a sum of dictinct terms of S. (Having distinct terms means having distinct indices, so that the values need not be distinct.) Call a sequence S *complete* if $P(S)$ contains all sufficiently large integers. Often writers have called S complete only if $P(S)$ contains all positive integers; we will call such a sequence *entirely complete*. Considerable study, spanning thirty years, has been devoted to completeness and related properties. (See [1; 2, Chap. 6] for surveys of the subject.)

It is a commonplace observation that completeness is not a very "robust"

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property, in that removing a few terms of a complete sequence can often destroy the property. Entire completeness is even less robust. Therefore, it is often more interesting to consider the following property. Call a sequence S *strongly complete* if it remains complete after removal of any finite number of terms.

Although strong completeness is a very interesting property and will figure considerably in what follows, we will be primarily concerned with another notion of robust properties, namely that of properties that are preserved under perturbation. If $S = \{s_n\}$ and $X = \{x_n\}$ are sequences, say that a sequence of integers $T = \{t_n\}$ is an X-perturbation of S if for every positive n, t_n lies between s_n and $s_n + x_n$. Note that this definition allows the x_n to be negative or zero.

Completeness is too restrictive a poperty to be stable under X-perturbation, unless X contains some zeros. Hence, following Folkman [3], we say that S is *subcomplete* if $P(S)$ contains any infinite arithmetic progression. This property does lead to interesting results. In particular, Burr [4] has shown that if $x_n = O(n^\alpha)$, where $\alpha < 1$, then any X-perturbation of the values of any polynomial of degree at least one is subcomplete. In fact, this holds even for "polynomials" with non-integral exponents.

From this, the question naturally arises whether the above could hold for $\alpha \geqslant 1$. We have not been able to deal with the case $\alpha = 1$, but we will show that if $\alpha > 1$ and the early terms of X are sufficiently large, then any sequence is X-perturbable into a sequence that is not subcomplete. To show this will be the primary task of Section 2 of this paper; actually somewhat more will be shown.

From the above, subcompleteness is a somewhat "fragile" property. It is perhaps remarkable that, on the other hand, incompleteness is far more fragile. In Section 3, we will show that any sequence satisfying rather mild growth conditions can be very slightly perturbed into a sequence which is strongly complete. Section 3 will also explore the limits of this phenomenon.

In the following sections, lower-case letters will denote integers, upper-case letters will denote sequences of integers, and Greek letters will denote real numbers.

2. Perturbations That Destroy Completeness Properties

Clearly, any sequence is 1-perturbable into a sequence which is not complete, since the perturbed sequence can be made to consist of even numbers only. Because of this, the interesting questions of this type center on subcompleteness. Our principal result in this section is the following.

THEOREM 2.1. *If the sum*

$$\sum_{n=1}^{\infty} \frac{1}{|x_n|}$$

is sufficiently small, and if $X = \{x_n\}$, then any sequence has an X-perturbation which is not subcomplete.

We will prove this result in considerably stronger form, but first we prove two lemmas. The following result, of interest in itself, is based on an idea of Cassels (see [5, Lemma 9; 6, Lemma 2]). Write $\|\alpha\|$ for the distance from α to the nearest integer.

LEMMA 2.1. *Let $S = \{s_1, s_2, \ldots\}$ be an infinite sequence of integers, and suppose that for some irrational α it happens that*

$$\sum_{i=1}^{\infty} \|\alpha s_i\| = \gamma < \tfrac{1}{2}.$$

Then the density of $P(S)$ in any infinite arithmetic progression is no more than 2γ.

Proof. We first note that if n is any integer satisfying $\|\alpha n\| \geqslant \gamma$, then $n \notin P(S)$; for suppose, on the contrary, that

$$n = \sum_{i \in I} a_i.$$

But then

$$\|\alpha n\| \leqslant \sum_{i \in I} \|\alpha a_i\| < \gamma \leqslant \|\alpha n\|,$$

a contradiction.

Therefore, the lemma will be proved if for every infinite progression $A = \{a + b, 2a + b, 3a + b, \ldots\}$, the density of n in A for which $\|\alpha n\| < \gamma$ is equal to 2γ. But this is an immediate consequence of the fact that the fractional parts of the sequence $\{\alpha a, 2\alpha a, 3\alpha a, \ldots\}$ are uniformly distributed in the unit interval, completing the proof.

Our next lemma is surely well known, although we know of no explicit reference. It follows from basic results in Diophantine approximation, so we will not include a proof.

LEMMA 2.2. *There exist real numbers α and δ with the property that any m consecutive integers contain an n for which $\|\alpha n\| < \delta/m$.*

In the above, α can be taken to be any quadratic irrationality, or any real number whose continued fraction has bounded partial quotients. If we make the choice $\alpha = (1+\sqrt{5})/2$, δ can be taken to be 2, and even somewhat smaller.

The next theorem clearly includes Theorem 2.1.

THEOREM 2.2. *Let α and δ be as in Lemma 2.2, and let $X = \{x_n\}$ satisfy*

$$\sum_{n=1}^{\infty} \frac{1}{|x_n|} < \frac{1}{2\delta}.$$

Then any sequence is X-perturbable into a sequence S for which the density of $P(S)$ in any infinite arithmetic progression is less than 1.

Proof. By Lemma 2.2, the perturbed sequence $\{s_n\}$ can be made to satisfy $\|\alpha s_n\| < \delta/|x_n|$, and hence

$$\sum_{n=1}^{\infty} \|\alpha s_n\| < \tfrac{1}{2}.$$

The result now follows from Lemma 2.1.

We close this section with an immediate corollary to Theorem 2.1. Call a sequence S *strongly subcomplete* if it remains subcomplete after removal of any finite number of terms.

THEOREM 2.3. *If the sum*

$$\sum_{n=1}^{\infty} \frac{1}{|x_n|}$$

converges, and if $X = \{x_n\}$, then any sequence has an X-perturbation which is not strongly subcomplete.

The results of this section, together with those of [4], still leave open many questions, some of which will be discussed in Section 4.

3. PERTURBATIONS THAT PRODUCE COMPLETENESS PROPERTIES

We will see that, generally speaking, very slight perturbations suffice to render sequences strongly complete, in contrast to the results in Section 2. We begin by stating two of the primary results in this section.

THEOREM 3.1. *Let $S = \{s_n\}$ satisfy $s_{n+2} \leqslant 2s_n$ for large n. Then S is* 1-*perturbable into a strongly complete sequence.*

THEOREM 3.2. *Let $S = \{s_n\}$ satisfy $s_{n+3} \leqslant 2s_n$ for large n, and let X have infinitely many nonzero terms. Then S is X-perturbable into a strongly complete sequence.*

We will defer the proofs of these results. Note that the import of Theorem 3.2 is that the perturbations can take place at arbitrarily sparse points, given the stronger condition on S. Another difference between Theorems 3.1 and 3.2 is that in the latter, the perturbations can be required to be of either sign. It is not clear whether this distinction is actually relevant, but various facts, expecially Lemma 3.4 and Theorem 3.7, suggest that it is. We will now work toward proving Theorem 3.1, beginning with some definitions.

Call S *precomplete* if $P(S)$ contains arbitrarily long sequences of consecutive integers, and *strongly precomplete* if it remains precomplete upon removal of any finite number of terms. If $s_{n+1} - \sum_{i=1}^{n} s_i \leqslant b$ for some b, say that S is a *Σ-sequence.* Also say that $P(S)$ has *gaps bounded by k* if any $k+1$ consecutive positive integers contains a member of $P(S)$. By Lemma 3.2, these two properties are essentially the same. Finally, if c is a constant, and $C = \{c, c, c, \ldots\}$, call a C-perturbation a c-perturbation.

If $\mathscr{P}$ is any property of sequences we say that $\mathscr{P}$ is *strong* if any sequence having the property continues to have the property after removing any finite number of terms. Examples of strong properties are those of being strongly (pre-) complete, being a Σ-sequence, and being infinite. An important principle that we will use is that any set of strong properties that implies completeness also implies strong completeness.

The first two lemmas that follow are taken from [7]; their proofs are very simple.

LEMMA 3.1. *Suppose S has two disjoint subsequences A and B, where A is precomplete and B is a Σ-sequence. Then S is complete.*

LEMMA 3.2. *If S is a Σ-sequence, then $P(S)$ has gaps bounded by some k.*

LEMMA 3.3. *If a sequence A of integers has infinitely many disjoint subsequences $A_1, A_2, \ldots$, where for each i, $P(A_i)$ contains two consecutive integers, then A is strongly precomplete.*

Proof. Obvious. (Note that there is no need for the A_i to be infinite sequences.)

LEMMA 3.4. *If A is a Σ-sequence, then A is 1-perturbable into a strongly precomplete sequence.*

Proof. By Lemma 3.3, it clearly suffices to show that A can be 1-perturbed into a sequence A' for which $P(A')$ contains two consecutive integers. We first note that if any two different subsets of A have the same sum, this is trivial, so we assume that all the subset sums of A are distinct.

By Lemma 3.2, there is some k such that $P(A)$ has gaps bounded by k. Suppose that for some n, it happens that

$$a_n \geqslant \sum_{i=1}^{k-1} a_{n-i}. \tag{1}$$

Let m be the largest $m \in P(A)$ satisfying $m < a_n$. By assumption, m satisfies $a_n - k - 1 \leqslant m < a_n$. But by (1), m is the sum of at least k elements of A. Increasing $a_n - m - 1$ of these by 1, we create a sequence A' for which $a_n - 1$ and a_n are both in $P(A')$.

Consequently, we may assume that

$$a_n < \sum_{i=1}^{k} a_{n-i}$$

for all n. But this implies that $a_n = O(\alpha^n)$ for some $\alpha < 2$, and a simple counting argument shows that the subset sums of A cannot be all distinct. This contradiction completes the proof.

Our next result is interesting enough to be stated as a theorem.

THEOREM 3.3. *Suppose that $A = \{a_i\}$ and $B = \{b_i\}$ are both Σ-sequences. Then A can be 1-perturbed into a sequence $\{a'_n\}$ such that $\{a'_1, b_1, a'_2, b_2, \ldots\}$ is strongly complete.*

Proof. This is immediate from Lemmas 3.1 and 3.4.

LEMMA 3.5. *If a sequence $S = \{s_n\}$ satisfies $s_{n+1} \leqslant 2s_n$ for large n, then S is a Σ-sequence.*

Proof. Obvious.

Proof of Theorem 3.1. Obvious from Theorem 3.3 and Lemma 3.5.

We now work toward proving Theorem 3.2.

LEMMA 3.6. *Let $A = \{a_n\}$ be any infinite sequence and let $B = \{b_n\}$ be a Σ-sequence. Then A can be 1-perturbed (or (-1)-perturbed) into a sequence $\{s'_n\}$ such that $S = \{a'_1, b_1, a'_2, b_2, \ldots\}$ is strongly precomplete.*

Proof. We will show that we can construct S so that $P(S)$ contains two consecutive integers; the desired result then follows by induction, using Lemma 3.3. Since B is a Σ-sequence, $P(B)$ has gaps bounded by some k. Let

$m = a_1 + \cdots + a_k$. Then there is an $n \in P(B)$ for which $n-k-1 \leqslant m < n$. Increasing $n-m-1$ of the terms $a_1, \ldots, a_k$ by 1, we have our desired construction. The argument for (-1)-perturbations is exactly analogous.

As before, we state the following result as a theorem.

THEOREM 3.4. *Let $A = \{a_n\}$ be any infinite sequence, and let $B = \{b_n\}$ and $C = \{c_n\}$ be Σ-sequences. Then A can be 1-perturbed (or (-1)-perturbed) into a sequence $\{a_n'\}$ so that $\{a_1', b_1, c_1, a_2', b_2, c_2, \ldots\}$ is strongly complete.*

Proof. Immediate from Lemmas 3.1. and 3.6.

Proof of Theorem 3.2. By Lemma 3.5, S contains three disjoint Σ-sequences A, B, and C. Without loss of generality, $\{x_{3n}\}$ has infinitely many positive terms, and $A = \{s_{3n}\}$. Theorem 3.4 can now be applied using the appropriate subsequence of A. This completes the proof.

Theorems 3.1 and 3.2 point the way to studying sequences $S = \{s_n\}$, where $s_{n+1}/s_n \to \lambda$ as $n \to \infty$. These theorems then apply when $\lambda < 2^{1/2}$ and $\lambda < 2^{1/3}$, respectively. On the other hand, if $\lambda > 2$, a trivial counting argument shows that S cannot even be subcomplete. Thus there is a considerable amount of interesting unexplored territory in the exponential-growth case. Our next results will probe this territory to some extent. One important point is $\lambda = (1+\sqrt{5})/2$, and our next theorem relates to this. This result refines part of Theorem 5 of [6]; we will state it rather carefully, since all the conditions are best possible in some sense.

THEOREM 3.5. *Let $2 \leqslant s_1 \leqslant s_2$, and $s_n \geqslant s_{n-1} + s_{n-2}$ for $n \geqslant 3$. Then $S = \{s_1, s_2, \ldots\}$ is not complete.*

Proof. Form the sequence $\{t_n\}$ by setting $t_n = s_n + s_{n-2} + \cdots$, where the last term in the sum is either s_1 or s_2. Write this final term as s_ε. We will prove by induction that $t_{2n-1} - 1 \notin P(S)$ for any $n \geqslant 1$. This certainly holds for $n = 1$. Observe that

$$\begin{aligned} s_{n+1} - t_n &= s_{n+1} - s_n - s_{n-2} - s_{n-4} - \cdots \\ &\geqslant s_{n-1} - s_{n-2} - s_{n-4} - \cdots \\ &\cdots \\ &\geqslant s_{\varepsilon+1} - s_\varepsilon \\ &\geqslant 2 \qquad \text{if } n \text{ is even} \\ &\geqslant 0 \qquad \text{if } n \text{ is odd}. \end{aligned} \tag{1}$$

Suppose that $t_{2n-3} - 1 \notin P(S)$; we will prove that $t_{2n-1} - 1 \notin P(S)$. Let s_m

be the largest term in the (assumed) representation of $t_{2n-1}-1$. By (1), we have $s_{2n}-(t_{2n-1}-1)\geqslant 1$, so that $m<2n$. On the other hand,

$$\begin{aligned}(t_{2n-1}-1)-(s_{2n-2}+s_{2n-3}+s_{2n-4}+\cdots+s_1)\\ &=(s_{2n-1}-s_{2n-2}-s_{2n-4}-\cdots)-1\\ &=s_{2n-1}-t_{2n-2}-1\\ &\geqslant 1,\end{aligned}$$

where we have again used (1). Therefore, $m>2n-2$, so $m=2n-1$. But then $(t_{2n-1}-1)-s_m=t_{2n-3}-1\notin(S)$, so that in fact $t_{2n-1}-1\notin P(S)$. This completes the proof.

From this we have immediately that if $\lim_{n\to\infty}(s_{n+1}/s_n)=\lambda>(1+\sqrt{5})/2$, then S cannot be strongly complete. When $\lambda=(1+\sqrt{5})/2$, the situation is delicate. Let f_n be the nth Fibonacci number. Then $\{f_n-c\}$ is not strongly complete for any $c\geqslant 0$, by Theorem 2.5. On the other hand, it is not difficult to see that $\{f_n+c\}$ is strongly complete for any $c>0$. Even more remarkable is the fact, due to Graham [8], that $\{f_n-(-1)^n\}$ is strongly complete, but becomes incomplete upon removal of *any* infinite subsequence. We state one consequence of Theorem 3.5 in a formal manner; note the strong contrast with Theorems 3.1 and 3.2.

THEOREM 3.6. *For any $\beta<(1+\sqrt{5})/2$, set $X=\{[\beta^n]\}$. Then there is a sequence $S=\{s_n\}$ satisfying $\lim_{n\to\infty}s_{n+1}/s_n=(1+\sqrt{5})/2$ such that no X-perturbation of it is complete.*

Proof. Set $S=\{f_n-2[\beta^n]\}$, and omit enough terms so that the conditions of Theorem 3.5 are satisfied.

Another property of some interest is that of being precomplete. In our next theorem we will show that in Lemma 3.4, the fact that the perturbations are positive is essential by exhibiting for each $c>0$ a Σ-sequence which is not $(-c)$-perturbable into a precomplete sequence.

THEOREM 3.7. *The sequence $S=\{k,2k,4k,8k,\dots\}$ is not $(2-k)$-perturbable into a precomplete sequence if $k\geqslant 3$.*

Proof. We will prove more, that no $(2-k)$-perturbation S' of S can have the property that $P(S')$ contains two consecutive terms. Let m and n be any two different numbers in $P(S)$, and let m' and n' be the values of the corresponding subset sums in S'. Without loss of generality, $m-n\geqslant k$. We will show that $m'-n'\geqslant 2$.

Consider the binary representations of m/k and n/k. If $m/k-n/k=1$, it is easily seen that the representaton of m/k has exactly one 1-bit that the representation of n/k lacks. In other words, the representation of m as a sum of terms of S has only one term missing from the corresponding representation of n. But even if this term is perturbed and all others left alone, we still have $m' - n' \geqslant 2$. By induction, $m' - n' \geqslant 2(m-n)/k$ in general, and the theorem is proved.

We close this section with one final result not directly related to perturbations. By Theorem 3.5, if $S = \{s_n\}$ satisfies $\lim_{n\to\infty} s_{n+1}/s_n \to \lambda > (1+\sqrt{5})/2$, S is not strongly complete. On the other hand, the following theorem exhibits a strongly complete S wich satisfies $\lim_{n\to\infty} s_n^{1/m} = 2$. Thus, somewhat surprisingly, the relatively smooth behavior of S is crucial to Theorem 3.5, and undoubtedly to some of the other results of this section as well.

THEOREM 3.8. *Let S be a sequence containing all sufficiently large powers of* 2, *and any infinite sequence of odd integers. Then S is strongly complete.*

Proof. Since the assumptions about S are strong, it suffices to prove that S is complete. Let A be the sequence of powers of 2 in S and let B be the sequence of odd numbers. Suppose that 2^k and all higher powers of 2 are in S. Then all numbers of the form $n \cdot 2^k$ are in $P(A)$. On the other hand, $P(B)$ certainly contains a complete sequence of residues (mod 2^k). From these two facts, it is clear that S is complete.

4. OPEN PROBLEMS

The results presented here suggest many interesting questions. An obvious such question is whether the condition on X can be weakened in Theorems 2.1 and 2.2, or at least in Theorem 2.3. It seems possible that the α in the proof might be made to depend on S, yielding such a weakening, perhaps to mere convergence of $\sum 1/|x_n|$. However, a weakening to $x_n = O(n)$, for instance, would probably require a completely new approach, if such a result were true at all. Of course the results of [4] or even [3] show that Theorems 2.1–2.3 are false for $x_n = O(n^{1-\epsilon})$, $\varepsilon > 0$. In the other direction, perhaps those results could be improved, but already the proof of the main theorem in [3], on which [4] is based, is very difficult.

Another question along the lines of [4] is the following: Does there exist a sequence S which grows faster than any polynomial and has the property that if $x_n = O(x^{1-\epsilon})$, then any $\{x_n\}$-perturbation of S is subcomplete? Using the results of [4] and Lemma 2.2 of [3], it is easy to show the following: Given such a sequence $X = \{x_n\}$, there is a sequence S which grows like $e^{x/\log x}$ (say), such that any X-perturbation of S is subcomplete. If S has to be chosen first, however, we do not know what to do. One can also ask similar questions with the property of subcompleteness replaced by that of $P(S)$ having density 1 in some arithmetic progression. In view of Theorem 2.2, this distinction is only relevant when the order of growth of x_n is close to n.

Section 3 is also a rich source of open problems, and we will mention a few in general terms. Problems involving sequences $S = \{s_n\}$ for which $s_{n+1}/s_n \to \lambda$ have already been discussed. In particular, are $\lambda = 2^{1/2}$ and $\lambda = 2^{1/3}$ actually critical in Theorems 3.1 and 3.2, respectively? It seems likely that the critical λ for Theorem 3.1 is either $\sqrt{2}$ or $(1+\sqrt{5})/2$, not in between.

Another question that has been mentioned is that of the extent to which smooth growth is important in the results. It almost certainly has some importance, in view of Theorems 3.5 and 3.8. Also, the significance of the sign of the perturbations is uncertain, since Lemma 3.4 and Theorem 3.7 do not clarify the situation much.

Finally, what can one say in Section 3 if completeness is replaced by subcompleteness? For example, does Theorem 3.6 hold for subcompleteness? Taking a more general point of view, does there exist a subcomplete sequence which grows more slowly than $\{2^n\}$ and which is not 1-perturbable into a complete sequence? It seems likely that such a sequence does exist; of course, if the growth condition is removed, $\{2, 4, 8, \ldots\}$ is such a sequence.

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