ROTATIONS OF THE CIRCLE

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The ideas presented here having arisen from the consideration of the following problem of Erdős.

Let T be the unit circle and suppose S_1 and S_2 are subsets of T such that for each i, $i = 1,2$, there is an infinite subset R_i of T so that the sets rS_i, where $r \in R_i$ are pairwise disjoint. Is it true that the inner measure of $S_1 \cup S_2$ is zero?

The answer to this question is yes and we shall present two solutions to this problem. Neither solution is difficult. But, each seems to lead to some interesting problems which will be formulated here.

Our first solution which is contained in the following theorem is based on the fact that the circle group is amenable.

THEOREM 1. Let G be a locally compact T_2 group with left invariant Haar measure λ and such that G admits an invariant mean. Let $S_1,\dots,S_k$ be subsets of G such that for each i, $1 \le i \le k$, there is an infinite subset R_i of G so that the sets rS_i, where $r \in R_i$, are pairwise disjoint and $\overline{R}_i$ is compact. Then the inner Haar measure of $\bigcup_{i=1}^{k} S_i$ is zero.

PROOF. Since G admits an invariant mean, there is a nonnegative, finitely additive extension μ, of λ to all subsets of G which is also left invariant.

It is enough to prove the theorem under the assumption that the sets S_i are pairwise disjoint.

Assume $\bigcup S_i$ has positive inner measure. Let F be a compact set $F \subset \bigcup S_i$, with $\lambda(F) > 0$. Let $D_i = F \cap S_i$. Now, the sets rD_i, where $r \in R_i$ are pairwise disjoint and $\bigcup\{rD_i : r \in R_i\}$ is a subset of the compact set $\overline{R}F$. Therefore, for each i, $\mu(D_i) = 0$.

This means $\mu(\bigcup D_i) = 0 = \mu(F) = \lambda(F) > 0$. This contradiction establishes the theorem.

Next we give an example to show that the conclusion of this theorem may be false if the group is not amenable.

Example. Let G be the orthogonal group on E^3. Notice that G acts transitively on S, the unit sphere of E^3. Let N be the north pole of S and let H be the stability subgroup of G at N. $H = \{g \in G : g(N) = N\}$. Then H is a closed subgroup of the compact group G. Let θ be the one-to-one map of the left coset space G/H onto $S : \theta(gH) = g(N)$.

Now, according to Hausdorff

$$S = A \cup B \cup C \cup D,$$

where A,B,C, and D are disjoint, D is countable and (*)$A \cong B \cup C$, $A \cong B$, $A \cong C$.

Let $E = \bigcup\{\theta^{-1}(d) : d \in D\}$. Since $\lambda(H) = 0$ and D is countable, $\lambda(E) = 0$. Let $S_1 = \bigcup\{\theta^{-1}(a) : a \in A\} S_2 = \bigcup\{\theta^{-1}(t) : t \in B \cup C\}$. Then $S_1 \cup S_2$ is a G_δ set with $\lambda(S_1 \cup S_2) > 0$. Also, because of (*), there are infinitely many pairwise disjoint translates of S_1 and S_2.

Thus, the first method of proof leads to the following problem.

PROBLEM. Let G be a locally compact T_2 group so that the conclusion of Theorem 1 holds. Is there a finitely additive left invariant extension of Haar measure to all subsets of G?

Before giving the second method of proof, let us state the following lemma.

LEMMA. Let G be a locally compact group with left invariant Haar measure λ. Let F be a compact set such that $0 < \lambda(F) < \infty$. For each positive integer n, there is a neighborhood V of e so that if $h_1, \ldots, h_n$ are points of V, then $\lambda(\bigcap\{h_i F : i \le n\}) > 0$.

The second method of proof is formulated for abelian groups.

THEOREM 2. Let G be a locally compact abelian group with Haar measure λ and let k be a positive integer. For each i, $1 \le i \le k$, let S_i and R_i be subsets of G such that R_i is infinite, $\bar{R}_i$ is compact, and the sets $S_i + g, g \in R_i$ are pairwise disjoint. The $\bigcup_{i=1}^{k} S_i$ has inner measure zero.

PROOF. Again, notice that we can and do assume that the sets S_i are disjoint. Let us assume that F is a compact set lying in $\bigcup S_i$ and $\lambda(F) > 0$.

Let $n > k$. According to the lemma, there is some neighborhood V of e so that if $h_p \in V, p = 1,\ldots,n^k$, then $\lambda(M) > 0$, where

$$M = F \cap (\cap\{F - h_p : p \le n^k\})$$

For each i, $1 \le i \le k$, obtain n+1 distinct points $g_{i0},\ldots,g_{in}$ of R_i so that for each k-tuple, $p = (p_1,\ldots,p_k)$ of the first n positive integers,

$$h_p = d_{1p_1} + \ldots + d_{kp_k}$$

is in V, where $d_{it} = g_{i0} - g_{it}$.

According to the lemma, there is some x in $F \cap (\cap\{F - h_p : p = (p_1,\ldots,p_k) \in \{1,\ldots,n\}^k)$.

For each i, let $M_i = \{p : x + h_p \in S_i\}$. The sets M_i are pairwise disjoint and each k-tuple of the first n-integers is in some M_i.

A contradiction will be reached by examining the cardinalities of the sets M_i. Notice that M_i has the following property. If $(p_1,\ldots,p_i,\ldots,p_k) \in M_i$ and $(r_1,\ldots,r_i,\ldots,r_k)$ is such that $r_j = p_j$, if $j \ne i$ and $r_i \ne p_i$, then $(r_1,\ldots,r_k) \notin M_i$. The reason for this is that if they were both in M_i, then $(x + d_{1p} + \ldots + d_{ip_i} + d_{kp_k}) + g_{ip_i} = x + d_{1p_1} + \ldots + g_{i0} + \ldots + d_{kp_k} = (x + d_{1r_1} + \ldots + d_{ir_i} + \ldots + d_{kr_p}) + g_{ir_k}$. Thus, the sets $S_i + g_{ip_i}$ and $S_i + g_{ir_i}$ would not be disjoint.

Now, because of this property of the sets M_i, we know that, $\text{card}(M_i) \le n^{k-1}$. Therefore,

$$n^k = \text{card}(\cup M_i) \le kn^{k-1}.$$

This contradiction establishes the theorem. Q.E.D.

Both of these proofs raise the question of estimating the size of a subset S of T which can be partitioned *into k sets each of which has n* pairwise disjoint rotations.

As Mycielski pointed out to us, one can use the extension of Haar measure to all subsets of T and argue along the lines of Theorem 1 to obtain the following estimate.

Theorem 3. If $S \subset T$ and $S \subset \cup\{A_i : i \le k\}$ where each set A_i has n pairwise disjoint rotations then the inner measure of S is $\le k/n$.

The proof of Theorem 2 was based on some simple combinational properties of finite sets. The problem we pose is that of estimating the size of a measurable subset M of T from the fact that M possesses two sets of rotations which avoid the contradiction obtained in Theorem 2. We formulate this as follows.

STATEMENT 1. Let $0 < \alpha$. There is a positive integer $n_0(\alpha)$ so that if M is a measurable subset of T with $\lambda(M) > \alpha$ and R_1 and R_2 are subsets of T with $|R_1|, |R_2| > n_0(\alpha)$, then there are points g_{1i} of R_1, g_{2i} of R_2, $i = 0,1,\ldots,4$ such that

$$M \cap (\cap\{M - h_p : p = (p_1,\ldots,p_3) \in \{1,\ldots,3\}^2\}) \neq \phi,$$

where

$$h_p = (g_{10}-g_{1p_1})+(g_{20}-g_{2p_2}).$$

In fact, statement 1 leads to the consideration of the following statement.

STATEMENT 2. Suppose $\alpha > 0$. There is a positive integer $t_0(\alpha)$ and a $\beta > 0$ so that if $M \subset T$, $\lambda(M) > \alpha$ and $R \subset T$, $|R| > t_0(\alpha)$, then there are points g_0, g_1, g_2, g_3 of R so that

$$\lambda\left(\bigcap_{i=0}^{3} M + (g_0 - g_i)\right) > \beta.$$

Clearly, statement 2 implies statement 1.

We have been unable to determine whether statement 2 is true. However, we have been led to the following statement.

STATEMENT 3. For each $c > 0$, there is an integer $\ell_0(c)$ and an integer $N_0(c)$ so that if $\ell > \ell_0(c)$ and $N > N_0(c)$ and

$$1 \leq a_1 < a_2 < \ldots < a_t \leq N \text{ where } t > cN,$$

then for each ℓ integers

$$b_1 < \ldots < b_\ell < N,$$

there is an arithmetic progession of three terms among the a's of difference some $b_j - b_i$.

At this time, we do not know which if any, of the preceding three statements is true.