Minimal Asymptotic Bases for the Natural Numbers

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The sequence A of nonnegative integers is an asymptotic basis of order h if every sufficiently large integer can be written as the sum of h elements of A. Let $M_k{}^A$ denote the set of elements that have more than one representation as a sum of h elements of A. It is proved that there exists an asymptotic basis A such that $M_k{}^A(x) = O(x^{1-1/h+\epsilon})$ for every $\epsilon > 0$. An asymptotic basis A of order h is *minimal* if no proper subset of A is an asymptotic basis of order h. It is proved that there does not exist a sequence A that is simultaneously a minimal basis of orders 2, 3, and 4. Several open problems concerning minimal bases are also discussed.

The set A of nonnegative integers is an *asymptotic basis of order h* if every sufficiently large integer can be represented as the sum of h elements of A. Many classical theorems of additive number theory are statements that a given sequence of integers is an asymptotic basis of some order. For example, Lagrange's theorem asserts that the squares $\{n^2\}_{n=0}^{\infty}$ form an asymptotic basis of order 4. Linnik [11] proved that the cubes $\{n^3\}_{n=0}^{\infty}$ form an asymptotic basis of order at most 7. Waring's problem is the conjecture, proved by Hilbert [10], that for every $k \geq 2$ there is a number $G(k)$ such that the sequence of kth powers $\{n^k\}_{n=0}^{\infty}$ is an asymptotic basis of order $G(k)$. Goldbach's conjecture that every large even number is the sum of two primes is equivalent to the assertion that the sequence of prime numbers is an asymptotic basis of order 3. Asymptotic bases in additive number theory have been widely investigated [7, 12, 16, 17].

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An asymptotic basis of order h is called *minimal* if no proper subset of A is an asymptotic basis of order h, that is, if each element of A is essential for the representation of infinitely many integers. Stöhr [17] introduced this concept of minimality. Härtter [8] gave a nonconstructive proof of the existence of minimal asymptotic bases, and Nathanson [13] constructed the first explicit examples. Nathanson [13, 15] also introduced the dual concept of maximal asymptotic nonbasis, and has considered multiplicative and combinatorial analogs of these additive number theoretical ideas. Minimal asymptotic bases and maximal asymptotic nonbases have been studied by Erdös, Härtter, Hennefeld, Nathanson, and Turjányi [2–6, 8, 9, 13–15, 18].

But many open problems remain. For example, does there exist a subsequence of the squares that is a minimal asymptotic basis of order 4? It is not even known if there exists a sequence A of squares that is an asymptotic basis of order 4 and satisfies $A(x) = O(x^{1/4})$. Does there exist an asymptotic basis A of order h such that, for every element $a \in A$, the set of numbers *not* representable as the sum of h elements of $A\backslash\{a\}$ has positive upper asymptotic density?

It is possible to construct an asymptotic basis of order h such that no subset of A is a minimal asymptotic basis of order h. Indeed, Erdös and Nathanson [6] have constructed an asymptotic basis A of order 2 consisting of square-free numbers with the property that, if F is any finite subset of A, then $A\backslash F$ is still an asymptotic basis of order 2, but, if I is any infinite subset of A, then $A\backslash I$ is an asymptotic nonbasis of order 2. The following problem is unsolved: If A is an asymptotic basis of order 2, then must A contain a subset that is a minimal asymptotic basis of order h for some $h \geqslant 2$?

Cassels [1] proved that, for every $h \geqslant 2$, there exists an asymptotic basis $A = \{a_n\}_{n=1}^\infty$ of order h such that $a_n = \alpha n^h + O(n^{h-1})$ for some $\alpha > 0$. Does there exist a minimal asymptotic basis $A = \{a_n\}_{n=1}^\infty$ of order h that satisfies Cassels's condition $a_n = \alpha n^h + O(n^{h-1})$?

In this paper we consider two open problems concerning minimal asymptotic bases. If the set A is an asymptotic basis of order h, then A is also an asymptotic basis of order k for every $k > h$. Similarly, if A is a minimal asymptotic basis of order k, and an asymptotic basis of order $h < k$, then A is certainly a minimal asymptotic basis of order h. But is it possible for a set A to be simultaneously a minimal asymptotic basis of two different orders? In particular, does there exist a minimal asymptotic basis of order 3 that is also an asymptotic basis of order 2? We prove below the weaker result that if A is an asymptotic basis of order 2, then A cannot be a minimal asymptotic basis of order 4.

Let $r_h{}^A(n)$ denote the number of representations of n as a sum of h elements of A, where representations differing only in the order of the summands are not counted separately. If A is a minimal asymptotic basis of order h, then

$r_h{}^A(n) \geq 1$ for all sufficiently large n, and $r_h{}^A(n) = 1$ for infinitely many n. Let $M_h{}^A = \{n \mid r_h{}^A(n) > 1\}$. If A is an asymptotic basis of order h, how "small" can $M_h{}^A$ be? Let $M_h{}^A(x)$ denote the number of elements of $M_h{}^A$ that do not exceed x. We shall construct, for every $h \geq 2$, an asymptotic basis A of order h such that $M_h{}^A(x) = O(x^{1-1/h+\epsilon})$ for every $\epsilon > 0$. In particular, there exists an asymptotic basis A of order 2 such that $M_2{}^A(x) = O(x^{1/2+\epsilon})$ for every $\epsilon > 0$. We conjecture that $\lim_{x\to\infty} M_2{}^A(x)/x^{1/2} = \infty$ for every asymptotic basis A of order 2, but we cannot prove that $\liminf_{x\to\infty} M_2{}^A(x)/x^{1/2} > 0$, or even that $\liminf_{x\to\infty} M_2{}^A(x)/x^{\alpha} > 0$ for some $\alpha > 0$.

Notation. If $A_0, A_1, ..., A_{h-1}$ are sets of integers, let $A_0 + A_1 + \cdots + A_{h-1}$ denote the set consisting of all sums of the form $a_0 + a_1 + \cdots + a_{h-1}$, where $a_i \in A_i$ for $i = 0, 1, ..., h-1$. If $A_0 = A_1 = \cdots = A_{h-1} = A$, denote $A_0 + A_1 + \cdots + A_{h-1}$ by hA. The set A is *periodic* if there exists an integer $m \geq 1$ such that $a + m \in A$ for all sufficiently large $a \in A$. Let $A(x)$ denote the number of positive integers in A that do not exceed x. Let $B\backslash A$ denote the relative complement of A in B.

LEMMA 1. *Let* $A = \{a_i\}_{i=1}^{\infty}$ *and let* $B_r = \{a_i + a_j\}_{i,j=r+1}^{\infty}$. *If* A *is not periodic, then* $\lim_{x\to\infty}(B_r(x) - A(x)) = \infty$.

Proof. Let $r < s < t$. Let $A_s = \{a_i + a_s\}_{i=s+1}^{\infty}$ and $A_t = \{a_i + a_t\}_{i=t+1}^{\infty}$. If $x \geq a_r + a_s$, then

$$B_r(x) \geq A_s(x) + (A_t\backslash A_s)(x) = A(x - a_s) - s + (A_t\backslash A_s)(x)$$
$$\geq A(x) - a_s - s + (A_t\backslash A_s)(x).$$

If $A_t\backslash A_s$ is finite, then $n \in A_t$ implies $n \in A_s$ for all sufficiently large n. Thus, if $a \in A$ is sufficiently large, then $a + a_t \in A_t$ and so $a + a_t \in A_s$, that is, $a + a_t = a' + a_s$ for some $a' \in A$, and so $a + (a_t - a_s) \in A$ for all sufficiently large $a \in A$. But $a_t - a_s \geq 1$, hence A is periodic. But this is false. Consequently, $A_t\backslash A_s$ is infinite, and so $\lim_{x\to\infty}(A_t\backslash A_s)(x) = \infty$. This proves the lemma.

LEMMA 2. *Let* A *be an asymptotic basis of order* h. *If* A *is periodic, then* A *is not minimal.*

Proof. Let A be a periodic asymptotic basis of order h. Then there exist positive integers m, p, and N, and finite sets $R \subseteq [0, m-1]$ and $F \subseteq [0, p]$ such that

$$A = F \cup \{a \equiv r \pmod{m} \mid r \in R, a > p\}$$

and such that $n \in hA$ for all $n > N$. Let $a = r + qm \in A$, where $a > m + p$ and $r \in R$. We shall show that $A \backslash \{a\}$ is an asymptotic basis of order h. Choose $n > ha + N$. Then $n \in hA$, and so

$$n = ka + a_1 + \cdots + a_{h-k}$$

where $k \geqslant 0$ and $a_i \in A \backslash \{a\}$ for $i = 1, \ldots, h - k$. If $k = 0$, then $n \in h(A \backslash \{a\})$. Suppose $k > 0$. Clearly, $k < h$ since $n > ha$. Since $n > ha + N$, it follows that $a_i > a > p$ for some a_i, say, $a_i = a_1$. Then $a_1 + km \equiv a_1 \equiv r$ (mod m) for some $r \in R$, and so $a_1 + km \in A$. Then

$$n = k(a - m) + (a_1 + km) + a_2 + \cdots + a_{h-k} \in h(A \backslash \{a\})$$

and so $A \backslash \{a\}$ is an asymptotic basis of order h. Thus, A is not minimal.

THEOREM 1. *Let A be an asymptotic basis of order* 2. *Then A is not a minimal asymptotic basis of order* 4.

Proof. Let $n \in 2A$ for $n \geqslant N$. Suppose that A is a minimal asymptotic basis of order 4. Fix $a^* \in A$. Then there is an infinite sequence of integers $n_1 < n_2 < \cdots$ such that $n_i \notin 4(A \backslash \{a^*\})$ for all $i = 1, 2, \ldots$. Fix n_i. Choose $b \in B = 2(A \backslash \{a^*\})$. Then $b = a_1 + a_2$, where $a_1, a_2 \in A \backslash \{a^*\}$. If $b \leqslant n_i - N$, then $n_i - b \geqslant N$ and so $n_i - b = a_3 + a_4$ for some $a_3, a_4 \in A$. Then $n_i = b + a_3 + a_4 = a_1 + a_2 + a_3 + a_4 \in 4A$. But $n_i \notin 4(A \backslash \{a^*\})$, and so $a_3 = a^*$ or $a_4 = a^*$, say, $a_4 = a^*$. Then $n_i - b = a_3 + a^*$. The number of integers of the form $n_i - b$ with $b \leqslant n_i - N$ is $B(n_i - N)$, and

$$B(n_i) - N \leqslant B(n_i - N) \leqslant A(n_i - a^*) \leqslant A(n_i)$$

and so $B(n_i) - A(n_i) \leqslant N$ for all n_i. Lemma 1 implies that the set A is periodic. But then Lemma 2 implies that A is not a minimal asymptotic basis of order 4. This proves the theorem.

The following lemma is well known.

LEMMA 3. *Let $s_1 < s_2 < \cdots$ be a sequence of integers such that $s_1 = 1$ and s_k divides s_{k+1} for $k = 1, 2, \ldots$. Then every positive integer n has a unique representation in the form $n = \sum_{k=1}^{\infty} d_k s_k$, where $0 \leqslant d_k < s_{k+1}/s_k$ and $d_k = 0$ for all but finitely many k.*

THEOREM 2. *For every $h \geqslant 2$, there exists an asymptotic basis A of order h such that, if M_h^A denotes the set of integers that have more than one representation as a sum of h elements of A, then $M_h^A(x) = O(x^{1-1/h+\epsilon})$ for every $\epsilon > 0$.*

Proof. Let $A = \bigcup_{i=0}^{h-1} A_i$, where A_i consists of all integers of the form

$$\sum_{k=1}^{\infty} c_k(k+1)^i (k!)^h,$$

where $0 \leqslant c_k \leqslant k$ and $c_k = 0$ for all but finitely many k. Since $(k!)^h$ divides $((k+1)!)^h$, Lemma 3 implies that every integer can be written uniquely in the form

$$\sum_{k=1}^{\infty} d_k(k!)^h,$$

where $0 \leqslant d_k < ((k+1)!)^h/(k!)^h = (k+1)^h$, and $d_k = 0$ for all sufficiently large k. But d_k can be written uniquely in the form

$$d_k = \sum_{i=0}^{h-1} c_{i,k}(k+1)^i,$$

where $0 \leqslant c_{i,k} \leqslant k$. Thus,

$$n = \sum_{k=1}^{\infty} d_k(k!)^h = \sum_{k=1}^{\infty} \sum_{i=0}^{h-1} c_{i,k}(k+1)^i (k!)^h$$

$$= \sum_{i=0}^{h-1} \left\{ \sum_{k=1}^{\infty} c_{i,k}(k+1)^i (k!)^h \right\} \in A_0 + A_1 + \cdots + A_{h-1} \subseteq hA.$$

Thus, A is an asymptotic basis of order h, and every positive integer has a unique representation in the form $n = a_0 + a_1 + \cdots + a_{h-1}$, where $a_i \in A_i$ for $i = 0, 1,..., h-1$.

Let $n \in M_h{}^A$. Then $r_h{}^A(n) > 1$, and so n can be written in the form $n = b_1 + b_2 + \cdots + b_h$, where $b_i \in A$ for $i = 1,..., h$, and there exists $t \in \{0, 1,..., h-1\}$ such that $b_i, b_j \in A_t$ for some $i \neq j$. Consequently, if $r_h(n) > 1$, then $n \in 2A_t + A_{i_1} + A_{i_2} + \cdots + A_{i_{h-2}}$, where $i_1,..., i_{h-2} \in \{0, 1,..., h-1\}$. We now estimate the number of integers of this form. If $(k!)^h \leqslant x < ((k+1)!)^h$, then

$$A_i(x) \leqslant \prod_{j=1}^{k} (j+1) = (k+1)! \leqslant (k+1) \, x^{1/h} = O(x^{1/h+\epsilon})$$

and so $(A_{i_1} + \cdots + A_{i_{h-2}})(x) = O(x^{1-2/h+\epsilon})$. The sumset $2A_t$ consists of all numbers of the form

$$\sum_{k=1}^{\infty} (c_k + c_k')(k+1)^t (k!)^h = \sum_{k=1}^{\infty} c_k''(k+1)^t (k!)^h,$$

where $0 \leqslant c_k^* \leqslant 2k$. If $(k!)^h \leqslant x < ((k+1)!)^h$, then

$$(2A_i)(x) \leqslant \prod_{j=1}^{k} (2j+1) = O(x^{1/h+\epsilon})$$

and so $(2A_i + A_{i_1} + \cdots + A_{i_{k-2}})(x) = O(x^{1-1/h+\epsilon})$. It follows that $M_h{}^A(x) = O(x^{1-1/h+\epsilon})$. This proves the theorem.

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