Problems and Results in Graph Theory and Combinatorial Analysis

PAUL ERDÖS

I published several papers with similar titles. One of my latest ones [13] (also see [16] and the yearly meetings at Boca Raton or Baton Rouge) contains, in the introduction, many references to my previous papers.

I discuss here as much as possible new problems, and present proofs in only one case. I use the same notation as used in my previous papers. $G^{(r)}(n; l)$ denotes an r-graph (uniform hypergraph all of whose edges have size r) of n vertices and l edges. If $r = 2$ and there is no danger of confusion. I omit the upper index $r = 2$. $K^{(r)}(n)$ denotes the complete hypergraph $G^{(r)}(n; \binom{n}{r})$. $K(a, b)$ denotes the complete bipartite graph ($r = 2$) of a white and b black vertices. $K_l^{(r)}(t)$ denotes the hypergraph of lt vertices $x_i^{(j)}$, $1 \leq i \leq t$, $1 \leq j \leq l$, and whose $\binom{l}{r}t^r$ edges are $\{x_{i_1}^{(j_1)}, \ldots, x_{i_r}^{(j_r)}\}$ where all the i's and all the j's are distinct. $e(G(m))$ is the number of edges of $G(m)$ (graph of m vertices), the girth is the length of a smallest circuit of the graph.

1

Hajnal, Szemerédi, and I have the following conjecture: There is a function $f(m)$ tending to infinity so that if $G(n)$ is a graph of n vertices and if every subgraph $G(m)(1 \leq m \leq n)$ of it can be made bipartite by the omission

ISBN 0-12-114350-3

of at most $f(m)$ edges, then $\chi(G(n)) \leq 3$ (or, in a slightly weaker form, its chromatic number is bounded). It is surprising that we have been able to make no progress with this extremely plausible conjecture.

It seems very likely that $f(m)$ can be taken as $c \log m$. Gallai [21] constructed a four chromatic graph $G(n)$ the smallest odd circuit of which has length at least $\sqrt{n}$, and it is easy to see that every subgraph $G(m)$ of this graph can be made bipartite by omitting $\sqrt{m}$ edges. Thus $f(m) \geq \sqrt{m}$.

Gallai and I conjectured that for every r there is an r-chromatic graph $G(n)$ the smallest odd circuit of which has length at least $n^{1/(r-2)}$. By a refinement of a method used by Hajnal and myself [17] (using Borsuk's theorem) Lovász proved this conjecture during this conference. The result of Lovász will imply that our $f(n)$ (if it exists) must be $o(n)$.

I stated that I can prove that Gallai's theorem [21] is best possible; that is, that if $G(n)$ is such that all odd circuits of $G(n)$ are longer than $cn^{1/2}$ then $\chi(G(n)) \leq 3$. This is almost certainly correct, but I have not been able to reconstruct my proof (which very likely was not correct). In fact I cannot even prove it if $cn^{1/2}$ is replaced by εn.

Another problem of Hajnal, Szemerédi, and me states: Let $\chi(G) = \aleph_1$. Is it true that, for every c, G has a subgraph of m vertices which cannot be made bipartite by the omission of cm edges? On the other hand we believe that for every $\varepsilon > 0$ there is a G with $\chi(G) = \aleph_1$ so that for every $m < \aleph_0$ every subgraph of G of m vertices can be made bipartite by the omission of fewer than $m^{1+\varepsilon}$ edges.

2

Let $|S| = n$, $A_k \subset S$, $|A_k| = cn$, $1 \leq k \leq n$. Denote by $f(n; c, \varepsilon)$ the largest integer so that there are at least $f(n; c, \varepsilon)$ sets, any pair of which have more than εn elements in common. It easily follows from Ramsey's theorem that, for every $i > 0$ and sufficiently small ε, $f(n; c, \varepsilon) > Cn^{1/(c+1)}$. I would like to prove

$$f(n; c, \varepsilon) > Cn^{1/2} \tag{1}$$

but I cannot even disprove $f(n; c, \varepsilon) > \eta n$. (1) would imply that every $G(n, [c_1 n^2])$ has $c_2 n^{1/2}$ vertices, any pair of which can be joined by vertex disjoint paths of length at most two. I can prove this with two replaced by four.

The following question is also of interest. Define a graph of n vertices as follows: The vertices are the n sets A_k. Join two of them if they have fewer than εn elements in common.

Estimate from above and below the largest possible value of the chromatic number of this graph. (1) would clearly follow if the chromatic number were less than $cn^{1/2}$.

The following question can now be posed: Let $|S| = 2n + k$. Define $G_{(n,k,l)}$ as follows: Its vertices are the $\binom{2n+k}{n}$ n-tuples of S. Two n-tuples A_i and A_j are joined if $|A_i \cap A_j| \leq l$. Estimate or determine $h(n; k, l) = \chi(G_{(n,k,l)})$.

A well-known conjecture of Kneser stated $h(n; k, 0) = k + 2$. This conjecture was recently proved by Bárány and Lovász.

A further complication can be introduced as follows: Let $h_m(n; k, l)$ be the largest integer for which $G_{(n,k,l)}$ has a subgraph $G(m)$ with $\chi(G(m)) = h_m(n; k, l)$. Determine or estimate $h_m(n, k, l)$.

The papers of Lovász and Bárány will appear soon.

3

A graph G_1 is said to be a *unique subgraph* of G if G_1 is a uniquely induced (or spanned) subgraph of G. Entringer and I [7] proved that there is a $G(n)$ which has more than

$$2^{\binom{n}{2}} \exp -(Cn^{(3/2)+\varepsilon}) \tag{1}$$

unique subgraphs. (1) was improved to $2^{\binom{n}{2}} \exp -(cn \log n)$ by Harary and Schwenk [22]. Finally Brouwer [3] improved (1) to

$$\frac{2^{\binom{n}{2}}}{n!} e^{-Cn}. \tag{2}$$

(2) is not far from being best possible since Pólya proved that the number of nonisomorphic graphs of n vertices is $(1 + o(1))2^{\binom{n}{2}}/n!$.

I always assumed that (2) is best possible, except for the value of C, but during this conference I had a discussion with J. Spencer who thought it quite possible that there is a graph $G(n)$ which has more than

$$\varepsilon \frac{2^{\binom{n}{2}}}{n!} \tag{3}$$

unique subgraphs where $\varepsilon > 0$ is independent of n. I do not believe this to be true but could not disprove it and offer 100 dollars for a proof and 25 for a disproof of (3).

In trying to decide about (3) perhaps the following idea could be helpful. I had no time to think it over carefully and have to apologize to the reader if it turns out to be nonsense: Put $t_n = n \log n/\log 2$. Consider the random graph $G(n; \binom{n}{2} - t_n)$. Is it true that almost all of them have a positive fraction of their subgraphs as unique subgraphs? Is there at least one such graph with this property? More generally, determine or estimate the largest $l_n = l_n(c)$ for which there is a $G(n; l_n)$ which has more than $c2^{l_n}$ unique subgraphs, i.e., a positive proportion of the subgraphs is unique.

4

Some problems on random graphs. Is it true that almost all graphs $G(n; Cn)$ contain a path of length cn? ($c = c(C) > 0$.) I conjectured this in 1974 [15] but Szemerédi strongly disagreed. He believes that for every fixed C the longest path contained in almost all $G(n; Cn)$ is $o(n)$. On second thought I think Szemerédi may very well be right, but at the moment nothing is known.

Let $G(n; t_n)$ be a graph of n vertices and t_n edges. Is it true that for almost all such graphs the edge connectivity is equal to the minimum degree (or valency) of the graph? Rényi and I [18] proved this if $t_n < \frac{1}{2}n \log n + cn \log \log n$. In this case the minimum degree is almost surely bounded. But the conjecture should hold for a much more extended range of t_n and perhaps in fact for all values of t_n.

The probability method easily gives that for every $\eta > 0$, $0 < c < \frac{1}{2}$, and and $n > n_0(\eta, c)$ there is a $G(n; [n^{1+c}])$ which has no triangle and, for every $m > \eta n$, every induced subgraph $G(m)$ satisfies

$$(1 - \eta)n^{1+c}\left(\frac{m}{n}\right)^{1+c} < e(G(m)) < (1 + \eta)n^{1+c}\left(\frac{m}{n}\right)^{1+c}.$$

In other words the graph behaves like a random graph.

Does this remain true for $\frac{1}{2} \leq c < 1$. It certainly fails for graphs $G(n; cn^2)$. See [19, 24].

5

It is well known that every graph with "many" edges contains a large complete bipartite graph. To fix our ideas consider a $G(n; [n^2/4])$. It is easy to see that it contains a $K(r, s)$ for $s = (1 + o(1))n/2^r$ as long as $r2^r \leq (1 + o(1))n$. Is this result best possible? By a simple computation using the probability method Simonovits, Lovász, and I showed that this result is indeed best possible as long as $2^r = o(n/(\log n)^2)$. In other words we can prove that there is a $G(n; (n^2/4))$ which contains no $K(r, (1 + \varepsilon)n/2^r)$ as long as $2^r < \eta(\varepsilon)n/(\log n)^2$.

What happens if $2^r > c(n/(\log n)^2)$? More precisely: Let $G(n; [n^2/4])$ be any graph of n vertices and $[n^2/4]$ edges. For every r satisfying $r2^r < n$ consider the largest s for which our graph contains a complete bipartite $K(r, s)$. Put $f(G) = \max_r s/(n/2r)$ and let $A(n)$ be the minimum of $f(G)$ extended over all $G(n; [n^2/4])$. How does $A(n)$ behave? On the one hand it could be $1 + o(1)$ and on the other hand it could tend to infinity.

One could further ask: Colour the edges of $K(n)$ by two colours, and let $K(r, s)$ be the largest monochromatic bipartite graph for $r2^r \le n$. Define $f_1(G)$ and A_1 as before. Conceivably $A_1 > A$. It even could be that $A \to 1$, $A_1 \to \infty$. For $2^r = o(n/(\log n)^2)$, $s < (1 + o(1))n/2^r$ holds here too.

Similarly problems can be posed for $G(n; cn^2)$, $c \neq \frac{1}{4}$, but these can easily be formulated by the interested reader.

Every $G(n; [n^2/4])$ contains a $K(r, r)$ for $r2^r < \varepsilon n$ (i.e., if $r < (1 - \varepsilon)\log n/\log 2$) and the probability method gives that it does not have to contain a $K(l, l)$ for $l > (2 + \varepsilon)\log n/\log 2$. The exact order of magnitude will probably be difficult to determine. Exactly the same question occurs if we colour a $K(n)$ by two colours and ask for the largest monochromatic $K(l, l)$. This difficulty is of course familiar in Ramsey theory.

See [23].

6

It is well known that for $n > n_0(\varepsilon, t)$ every $G^{(3)}(n; [\varepsilon n^3])$ contains a $K_3(t, t, t)$. Is it true that a $G^{(3)}(3n; n^3 + 1)$ contains a $G^{(3)}(9; 28)$? In particular does it contain a $K_3(3, 3, 3)$ and one more triple? (This conjecture is stated incorrectly in [13, p. 11].)

Very little is known about extremal problems on r-graphs (uniform hypergraphs where the edges have size r). Turán's classical problem of determining $f(n; K^{(3)}(4))$ is of course still open. I recently offered 500 dollars—in Turán's memory—for the determination of (or an asymptotic formula for) $f(n; K^{(3)}(4))$. As far as I know $f(n; G^{(3)}(4; 3))$ is also not yet known. The largest $G^3(n)$ without a $G^{(3)}(4; 3)$ I can construct it as follows: Put $n = a + b + c$ with a, b, c being as nearly equal as possible. Consider a $K_3(a, b, c)$ and put $a = a_1 + a_2 + a_3$, $b = b_1 + b_2 + b_3$, $c = c_1 + c_2 + c_3$ and iterate the preceding process. This gives a

$$G\left(n; (1 + o(1))\frac{n^3}{24}\right)$$

which contains no $G^{(3)}(4; 3)$ and which may be extremal.

Here I restate an old problem of mine which I consider very attractive. Let $G^{(r)}(n_i)$, $n_i \to \infty$, be a sequence of r-graphs of n_i vertices. We say that the family has subgraphs of edge density $\geq\alpha$ if, for infinitely many n_i, $G(n_i)$ has a subgraph $G(m_i)$, $m_i \to \infty$, so that $G(m_i)$ has at least $(\alpha + o(1))\binom{m_i}{r}$ edges. The theorem of Stone and myself implies that every $G^{(2)}(n; (n^2/2)(1 - (1/l) + \varepsilon))$ contains a subgraph of density $1 - [1/(l + 1)]$, and it is easy to see that this is best possible. If α is the largest such number then the $G(m_i)$ are called a family of subgraphs of maximal density.

Thus for $r = 2$ the possible maximal densities of subgraphs are of the form $1 - (1/l)$, $1 \leq l < \infty$. Now I conjecture that for $r > 2$ there are also only a denumerable number of possible values for the maximal densities α. I offer 500 dollars for the determination of these values for all $r > 2$—or also for a refutation of my conjecture.

The simplest unsolved problem here is as follows: Prove that there is an absolute constant $c > 0$ so that for every $\varepsilon > 0$ if

$$G^{(3)}\left(n_i; \left[\frac{n_i^{\,3}}{27}(1 + \varepsilon)\right]\right)$$

is a family of 3-graphs there is a family of subgraphs of edge density greater than $\frac{2}{9} + c$. I offer 250 dollars for a proof or disproof of this conjecture. Probably $(n^3/27)(1 + \varepsilon)$ can in fact be replaced by $(n^3/27) + n^{3-\eta_n}$ where $\eta_n \to 0$ as $n \to \infty$.

Similar unsolved problems on the possible maximal densities arise in multigraphs and digraphs as stated in a paper of Brown *et al.* [4].

Let $G^{(r)}(n)$ be an r-graph. Its edge graph is the ordinary graph (i.e., $r = 2$) whose vertices are the vertices of our $G^{(r)}(n)$. Two vertices in the edge graph are joined by an edge if they are contained in one of the r-tuples of $G^{(r)}(n)$. Simonovits and I observed that the edge graph of a $G^{(r)}(n; [(1 + \varepsilon)\binom{l}{r}n^r/l^r])$ contains a $K_{l+1}^{(r)}(t)$ for every $n > n_0(t)$. The edge graph of $K_l^r([n/l])$ shows that this result is essentially best possible.

See [1, 2, 5, 11, 20].

7

In a previous paper [14] I stated the following problem: "A problem in set theory led R. O. Davies and myself to the following question: Denote by $f(n, k)$ the largest integer so that if there are given in k-dimensional space n points which do not contain the vertices of an isosceles triangle, then they determine at least $f(n, k)$ distinct distances. Determine or estimate $f(n, k)$. In particular is it true that

$$\lim_{n\to\infty} \frac{f(n, k)}{n} = \infty? \tag{1}$$

(1) is unproved even for $k = 1$. Straus observed that if $2^k \geq n$ then $f(n, k) = n - 1$."

I perhaps should have given some explanation why (1) is difficult even for $k = 1$. Observe that $\lim_{n\to\infty} f(n, 1)/n = \infty$ implies Roth's theorem: $r_3 = o(n)$, where $r_3(n)$ is the smallest integer so that if $1 \leq a_1 < \cdots < a_k \leq n, k \geq r_3(n)$ then the a's contain an arithmetic progression (i.e., an isosceles triangle). The converse does not seem to be true, i.e., $r_3(n)/n \to 0$ does not imply (1).

Is it true that if $a_1, \ldots, a_m$ is a set of integers which does not contain an arithmetic progression of three terms then for $m > m_0(c)$ there are more than cm distinct integers of the form $a_j - a_i$? This is a special case of (1) if $k = 1$. At the moment I do not see how to prove this but perhaps it will not be very hard.

The following further questions are perhaps of some interest: Let there be given n points in k-dimensional space. Assume that every set of four of them determine at least five different distances. Is it then true that the n points determine at least $c_k n^2$ distinct distances. This is trivial for $k = 1$ since if four points on the line determine five distances three of the points must form an arithmetic progression and the fourth point is in general position. From this remark it easily follows that if there are n points on the line and any four determine at least five distances then the n points determine at least $(n^2/2) - cn$ distances. For $k > 1$ I do not know what happens, but again I am not sure if the question is really difficult.

Let $X_1, \ldots, X_n$ be n points in k-dimensional space. Denote by $A_k(X_1, \ldots, X_n)$ the number of distinct distances the n points determine. What is the set of possible values of $A_k(X_1, \ldots, X_n)$? I previously considered the smallest possible value of $A_k(X_1, \ldots, X_n)$ and, for $k > 1$, this is no doubt a very difficult question, but perhaps it is possible to make some nontrivial statement about the possible set of values of $A_k(X_1, \ldots, X_n)$, e.g., perhaps in a certain range it can take all values.

8

Let G be a graph of m edges. Denote by $f(m)$ the largest integer so that G always contains a bipartite graph of $f(m)$ edges. Edwards [6] and I proved that $f(m) > (m/2) + c\sqrt{m}$ and that in general this result is best possible.

Edwards in fact determined $f(m)$ explicitly.

Assume now that G has m edges and girth r (i.e., the smallest circuit of G has r edges). Denote by $f_r(m)$ the largest integer so that G always contains a bipartite graph of $f_r(m)$ edges. Lovász and I proved that

$$\frac{m}{2} + c_2 m^{1-c_r''} < f_r(m) < \frac{m}{2} + c_1 m^{1-c_r'}, \tag{1}$$

where c_r' and c_r'' are greater than $\frac{1}{2}$ and less than one and tend to one as r tends to infinity.

It seems certain that there is an absolute constant c_r so that

$$\frac{m}{2} + m^{c_r - \varepsilon} < f_r(m) < \frac{m}{2} + m^{c_r + \varepsilon}, \tag{2}$$

but at the moment we cannot prove (2). If (2) holds then the next step would be to get an asymptotic formula for $f_r(m) - (m/2)$.

Now we outline the proof of (1). The upper bound follows by the probability method and we do not give it here (an outline has already been published in Hungarian) since it is fairly standard. We prove the lower bound in some detail but for simplicity assume $r = 4$ (i.e., G has m edges and no triangle). The proof of the general case is not really different. We need two lemmas.

Lemma 1 *Let G have m edges and chromatic number k where $k = 2r$ or $2r - 1$. Then G contains a bipartite subgraph of at least $m(r/2r - 1)$ edges.*

If $m = \binom{s}{2}$ then $K(s)$ shows that the lemma is best possible.

Since G has chromatic number k one can decompose its vertex set into k independent subsets S_i, $1 \le i \le k$. Decompose the index set $\{i\colon 1 \le i \le k\}$ in all possible ways into two disjoint sets A_u, B_u, $A_u = [k/2]$, $B_u = [(k+1)/2]$. The number of these decompositions clearly equals $\binom{k}{[k/2]}$. Consider all the $\binom{k}{[k/2]}$ bipartite subgraphs of G_u, $1 \le u \le \binom{k}{[k/2]}$, where in G_u a vertex is white if it belongs to an S_i, $i \in A_u$, and is black otherwise. Every edge of G clearly occurs in exactly $2\binom{k-2}{[k/2]-2}$ of the bipartite graphs G_u. Thus if $e(G_u)$ denotes the number of edges of G_u we have

$$\sum e(G_u) = 2m\binom{k-2}{[k/2]-1}$$

or if $k = 2r$ or $2r - 1$

$$\max_u e(G_u) \ge m \cdot 2\binom{k-2}{[k/2]-1} = m\,\frac{r}{2r-1}$$

which proves the lemma. ∎

Lemma 2 *Let G have m edges and no triangle. Then its chromatic number is less than*

$$c_1\left(\frac{m \log\log m}{\log m}\right)^{1/3} = t_m.$$

Graver and Yackel proved that a graph of chromatic number t_m has at least

$$\frac{c_2 t_m^{\,2} \log t_m}{\log \log t_m}$$

vertices. Clearly each vertex can be assumed to have valency (or degree) at least t_m. Thus a t_m-chromatic graph has at least

$$\frac{c_2 t_m^{\,3} \log t_m}{\log \log t_m}$$

edges, which proves our lemma. ■

The lemma is not very far from being best possible. I showed that there is a graph of m edges which has no triangle and whose chromatic number is greater than $m^{1/3}/\log m$.

Lemmas 1 and 2 immediately give that

$$f_4(m) > \frac{m}{2} + cm^{2/3}\left(\frac{\log m}{\log \log m}\right)^{1/3}$$

which completes the proof of our theorem. The proof if $r > 4$ is clearly almost identical.

The upper bound given by the probability method for $f_4(m)$ is very much worse than $m^{2/3}$—we have no guess for the correct exponent.

See [9, 10].

9

In 1972 Faber, Lovász, and I stated the following conjecture: Let $|A_k| = n$, $1 \le k \le n$. Assume that any two of these sets have at most one element in common. Is it then true that one can colour the elements of $\bigcup_{k=1}^{n} A_k$ by n colours so that every set contains elements of all colours?

I consider this conjecture very attractive—also it does not seem to be easy and I offer 250 dollars for a proof or disproof.

Several mathematicians reformulated the conjecture as follows: Define $G(A_1, \ldots, A_n)$ as follows: The vertices are the elements of $\bigcup_{i=1}^{n} A_i$. Two vertices are joined if they belong to the same A_i. Prove $\chi(G(A_1, \ldots, A_n) = n$. A theorem of de Bruijn and myself implies that it contains no K_{n+1}.

One advantage of this definition is that one can easily ask new questions. Let $h(m)$ be the smallest integer for which there is a set system $A_1, \ldots, A_{h(m)}$ ($|A_i| = n$, $|A_i \cap A_j| \le 1$) for which $\chi(A_1, \ldots, A_{h(m)}) = m$.

Many further generalisations are possible, e.g., $|A_i \cap A_j| \le 1$ can be replaced by $|A_i \cap A_j| \le l$. Also $|A_i \cap A_j| \le 1$ can be further strengthened in various ways. We could assume, say, that among any three sets A_i, A_j, A_l at least two are disjoint, etc.

One more new problem: In all the cases I know $G(A_1, \ldots, A_{h(m)})$ contains a $K(m)$. That this must be the case is not hard to show for fixed n and m sufficiently large. Is it true for every n and m?

See [5a].

10

Tutte (and later independently Ungár, Zykov, and Mycielski) was the first to construct graphs of arbitrarily large chromatic number having no triangles. Then I [15] and later Lovász proved that for every r there is a graph of girth r and chromatic number k.

Now Hajnal and I asked: Is there an $A(k, r)$ so that every G with $\chi(G) \geq A(k, r)$ contains a subgraph of girth r and chromatic number k? Rödl recently proved that $A(k, 4)$ exists for every k, but his upper bound for $A(k, 4)$ is probably very poor. I think $A(k, 4) < ck$ is true. It would be very interesting if one could prove that $A(k, r)$ exists for every k and r and if we could obtain some knowledge of the order of magnitude of $A(k, r)$, e.g., is it true that $\lim_{k\to\infty} [A(k, r+1)/A(k, r)] = \infty$?

REFERENCES

1. B. Bollobás and P. Erdös, On the structure of edge graphs, *Bull. London Math. Soc.* **15** (1973) 317–321.
2. B. Bollobás, P. Erdös, and M. Simonovits, On the structure of the edge graphs II, *J. London Math. Soc.* **12** (1976) 219–224.
3. A. E. Brouwer, On the number of unique subgraphs of a graph, *J. Combinatorial Theory Ser. B* **18** (1975) 184–185.
4. W. G. Brown, P. Erdös, and M. Simonovits, Extremal problems for directed graphs, *J. Combinatorial Theory Ser. B* **15** (1973) 77–93.
5. W. G. Brown, P. Erdös, and V. T. Sós, Some extremal problems on r-graphs, *in* "New Directions in the Theory of Graphs (Proceedings of the Third Ann Arbor Conference on Graph Theory)" (F. Harary and E. M. Palmer, eds.), pp. 53–63. Academic Press, New York, 1973; see also Erdös and Spencer [20].

5a. N. G. de Bruijn and P. Erdös, On a combinatorial problem, *Indag. Mat.* **10** (1948) 421–423.

6. C. S. Edwards, An improved lower bound for the number of edges in a largest bipartite subgraph, *Recent Advances Graph Theory, Proc. Symp. Prague* (1974) 167–181; Some extremal properties of bipartite subgraphs, *Canad. J. Math.* **25** (1973) 475–485.
7. R. C. Entringer and P. Erdös, On the number of unique subgraphs of a graph, *J. Combinatorial Theory Ser. B* **13** (1972) 112–115.
8. P. Erdös, Graph theory and probability, *Canad. J. Math.* **11** (1959) 34–38.
9. P. Erdös, Graph theory and probability II, *Canad. J. Math.* **13** (1961) 346–352.
10. P. Erdös, On even subgraphs of graphs (in Hungarian), *Mat. Lapok* **18** (1964) 283–288.
11. P. Erdös, On some extremal problems on r-graphs, *Discrete Math.* **1** (1971) 1–6.

12. P. Erdös, On a problem of Grünbaum, *Canad. Math. Bull.* **15** (1972) 23–26.
13. P. Erdös, Problems and results in combinatorial analysis, *Accad. Naz. Lincei Colloq. Internat. Teorie Comb., Rome* **2** (1973) 3–17.
14. P. Erdös, Problems and results on combinatorial number theory, *in* "A Survey of Combinatorial Theory," pp. 117–136, 136. North-Holland Publ., Amsterdam, 1973.
15. P. Erdös, Some new applications of probability methods to combinatorial analysis and graph theory, *Proc. Southeastern Confer. Combinatorics, 5th, Graph Theory Comput., Boca Raton* (1974) 39–51.
16. P. Erdös, Problems and Results in Graph Theory and Combinatorial Analysis, *Proc. British Combinatorial Conf., 5th* (1975) 169–192.
17. P. Erdös and A. Hajnal, Kromatikus grafókról (in Hungarian), *Mat. Lapok* **18** (1967) 1–4.
18. P. Erdös and A. Rényi, On the strength of connectedness of a random graph, *Acta Math. Acad. Sci. Hungar.* **12** (1961) 261–267; see also "The Art of Counting," pp. 574–624. MIT Press, Cambridge, Massachusetts.
19. P. Erdös and J. Spencer, "Probabilistic Methods in Combinatorics," Academic Press, New York, 1974.
20. P. Erdös and A. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1946) 1087–1091.
21. T. Gallai, Kritische Graphen, *Publ. Math. Inst. Hungar. Acad. Sci.* **8** (1973) 165–192; see especially pp. 172–173, 186–189.
22. F. Harary and A. J. Schwenk, On the number of unique subgraphs, *J. Combinatorial Theory Ser. B* **15** (1973) 156–160.
23. T. Kövári, V. T. Sós, and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.* **3** (1954) 50–57; see also Erdös and Spencer [20].
24. L. Pósa, Hamiltonian circuits in random graphs, *Discrete Math.* **14** (1976) 359–364.

AMS 05C35

MATHEMATICS INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, HUNGARY