Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series A
All Rights Reserved by Academic Press, New York and London

Vol. 32, No. 2, March 1982
*Printed in Belgium*

# Minimal Decompositions of Hypergraphs into Mutually Isomorphic Subhypergraphs

F. R. K. CHUNG

*Bell Laboratories, Murray Hill, New Jersey 07974*

P. ERDÖS

*Mathematics Institute of the Hungarian Academy of Sciences, Budapest, Hungary*

AND

R. L. GRAHAM

*Bell Laboratories, Murray Hill, New Jersey 07974*

Received March 23, 1981

DEDICATED TO PROFESSOR MARSHALL HALL, JR., ON THE OCCASION OF HIS RETIREMENT

## INTRODUCTION

By an *r-uniform hypergraph* (or *r-graph*, for short) $H = (V, E)$ we mean a collection $E = E(H) = \{E_1, ..., E_m\}$ of $r$-element subsets (called *edges*) of a set $V = V(H)$, called the *vertices* of $H$. Let $\mathscr{H} = \{H_1, ..., H_k\}$ be a family of $r$-graphs, each having the same number of edges. By a *U-decomposition* of $\mathscr{H}$ we mean a set of partitions of the edge sets $E(H_i)$ of the $H_i$, say $E(H_i) = \sum_{j=1}^{t} E_{i,j}$, such that for each $j$, all the $E_{i,j}$ are *isomorphic* (as hypergraphs). Such decompositions always exist since (by our assumptions) we can always take all the $E_{i,j}$ to be single edges.

Let us define the quantity $U(\mathscr{H})$ as the least possible value of $t$ a $U$-decomposition of $\mathscr{H}$ can have. Finally, we let $U_k(n, r)$ denote the largest possible value $U(\mathscr{H})$ can assume as $\mathscr{H}$ ranges over all families of $k$ $r$-graphs, each having $n$ vertices and the same (unspecified) number of edges.

For the value $r = 2$, $r$-graphs are just ordinary graphs and in this case, the



0097-3165/82/020241-11\$02.00/0

Copyright © 1982 by Academic Press, Inc.
All rights of reproduction in any form reserved.

functions $U_k(n, 2) = U_k(n)$ have been investigated extensively by the authors and others in [1, 2]. In particular, it is known that

$$U_2(n) = \tfrac{2}{3}n + o(n),$$

and

$$U_k(n) = \tfrac{3}{4}n + o_k(n), \qquad k \geqslant 3.$$

In this paper we continue this study to the much more complex case of $r > 2$. Our basic results are the following (where $c_1, c_2, \ldots$, denote appropriate positive constants):

$$c_1 n^{4/3} \log\log n / \log n < U_2(n; 3) < c_2 n^{4/3}; \tag{1}$$

$$\text{for any } \varepsilon > 0, \quad c_3 n^{2-2/k-\varepsilon} < U_k(n; 3) < c_4 n^{2-1/k}; \tag{2}$$

$$c_5 n^{r/2} < U_2(n; r) < c_6 n^{r/2} \qquad \text{for } r \text{ even}; \tag{3}$$

$$c_7 n^{(r-1)^2/(2r-1)} < U_2(n; r) < c_8 n^{r/2} \qquad \text{for } r \text{ odd}; \tag{4}$$

$$n^{r-1-r/k} \leqslant U_k(n; r) \leqslant n^{r-1-1/k} \qquad \text{for } r \geqslant 3. \tag{5}$$

## Preliminaries

We first prove several auxiliary lemmas. Suppose $\mathscr{H} = \{H_1, \ldots, H_k\}$, where each $H_i$ is an $r$-graph having $n$ vertices and $e$ edges. Let us denote by $c(\mathscr{H})$ the maximum number of edges in any hypergraph $H$ occurring in all the $H_i$ as a common subhypergraph.

Lemma 1.

$$c(\mathscr{H}) \geqslant \frac{e^k}{\binom{n}{r}^{k-1}}.$$

*Proof.* Let $\Omega_i$ denote the set of all one-to-one mappings of $V(H_i)$ into $V(H_1)$. For $\lambda_i \in \Omega_i$, $e_i \in E(G_i)$, $1 \leqslant i \leqslant k$, define

$$\begin{aligned} I_{\lambda_2,\ldots,\lambda_k}(e_1, \ldots, e_k) &= 1 \qquad \text{if } \lambda_i \text{ maps } e_i \text{ onto } e_1, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

where we say that $\lambda_i$ maps $e_i$ onto $e_1$ if $e_1 = \bigcup_{x \in e_i} \lambda_i(x)$. Consider the sum

$$S = \sum_{\substack{e_1 \in E(H_1) \\ \vdots \\ e_k \in E(H_k)}} \sum_{\substack{\lambda_2 \in \Omega_2 \\ \vdots \\ \lambda_k \in \Omega_k}} I_{\lambda_2, \ldots, \lambda_k}(e_1, \ldots, e_k)$$

$$= \sum_{\substack{e_1 \in E(H_1) \\ \vdots \\ e_k \in E(H_k)}} (r!(n-r)!)^{k-1} = e^k (r!(n-r)!)^{k-1}.$$

Since $|\Omega_i| = n!$ for all $i$ then for some choice of $\bar{\lambda}_2 \in \Omega_2, \ldots, \bar{\lambda}_k \in \Omega_k$,

$$\sum_{\substack{e_1 \in E(H_1) \\ \vdots \\ e_k \in E(H_k)}} I_{\bar{\lambda}_2, \ldots, \bar{\lambda}_k}(e_1, \ldots, e_k) \geqslant \frac{S}{(n!)^{k-1}} = \frac{e^k}{\binom{n}{r}^{k-1}}.$$

Consequently, the $\bar{\lambda}_i$, $2 \leqslant i \leqslant k$, determine a subhypergraph $H$ common to all of the $H_i$ which has at least $e^k / \binom{n}{r}^{k-1}$ edges. ∎

Lemma 2. *Let $H$ be an r-graph with $|E(H)| \geqslant rab + 1$. Suppose* $\deg v = |\{e \in E(H): v \in e\}| < a$ *for all vertices $v \in V(H)$. Then $H$ contains $b$ disjoint edges.*

*Proof.* Suppose $F$ is a maximal set of disjoint edges. If $|F| < b$, the number of edges containing *some* element of $F$ must be at most $|F|\, ar < |E(H)|$, contradicting the maximality of $F$. ∎

Lemma 3. *If $r = 3$,*

$$c(\mathscr{H}) \geqslant \sqrt{\frac{e}{5n}}.$$

*Proof.* It suffices to prove there is a *star* with $t = \lceil \sqrt{e/5n} \rceil$ edges contained in each $H_i$. By a star $S$ we mean a collection of edges $e_i$ such that for some point $x$, $e_i \cap e_j = \{x\}$ for all $i \neq j$. Suppose $H$ has $n$ vertices and $e$ edges and does not contain $S$. Consider the set $P$ of disjoint pairs of vertices of $V(H)$ defined as follows:

(i) Select $v_1$ with $\deg_H(v_1) \geqslant \deg_H(v)$ for any $v \in V(H)$. Let $v_1^*$ be a vertex in $H_{x_1}$ of maximum degree and define $P_1 = \{v_1, v_1^*\}$ (where, for $x \in V(H)$, $H_x$ denotes the ordinary (2-) graph with edge set $\{\{y, z\}: \{x, y, z\} \in E(H)\}$).

(ii) Suppose now that $P_1, \ldots, P_i$ have been defined. We form $P_{i+1}$ as follows. Choose $v_{i+1}$ so that:

(a) $v_{i+1} \notin P_j$ for $1 \leqslant j \leqslant i$;

(b) $v_{i+1}$ has maximum degree in the subhypergraph induced on $V(H)-X_i$, where

$$X_i=\bigcup_{j=1}^{i} P_j.$$

Let $v_{i+1}^*$ be the vertex the graph $H_{v_{i+1}}-X_i$ having maximum degree at least one in $H_{v_{i+1}}$. Define $P_{i+1}\equiv\{v_{i+1},v_{i+1}^*\}$. We continue this process as long as possible. The final set of pairs $P$ is defined to be $\bigcup_{i\geqslant 1}P_i$.

Let $d_i$ denote the degree of $v_i$ in the hypergraph induced in $V(H)-X_i$ (with $X_0$ taken to be $\varnothing$). Since $H$ does not contain a copy of $S$, we have

$$|\{\bar{e}\in E(H)\colon P_i\subseteq\bar{e}\}|\geqslant d_i/2t$$

for all $i\geqslant 1$. Let $d_i^*$ denote the degree of $v_i^*$ in the hypergraph induced on $V(H)-X_{i-1}-\{v_i\}$. Then $d_i\geqslant d_i^*$ and $\sum_i(d_i+d_i^*)\geqslant e$. Therefore, $\sum_i d_i\geqslant e/2$.

Now, for any $v\in V(H)$, define $\alpha(v)$ by

$$\alpha(v)\equiv|\{\bar{e}\in V(H)\colon\bar{e}=\{v\}\cup P_i\text{ for some }i\}|.$$

Thus,

$$\begin{aligned}\sum_v\alpha(v)&\geqslant\sum_i|\{\bar{e}\in E(H)\colon P_i\in\bar{e}\}|\\&\geqslant\sum_i d_i/2t\\&\geqslant e/4t.\end{aligned}\tag{6}$$

However, the assumption that $S\nsubseteq H$ implies $\alpha(v)\leqslant t-1$. Therefore, by (6)

$$(t-1)n\geqslant e/4t,$$

which clearly contradicts the hypothesis that $t\leqslant\sqrt{e/5n}$. ∎

In a similar way we can prove the following.

LEMMA 4. *Let $\mathscr{H}$ be a family of r-graphs, each with e edges. Then*

$$c(\mathscr{H})\geqslant\sqrt{\frac{ce}{n^{r-2}}},$$

*where c is a constant depending on k.*

## BOUNDS ON $U_2(n; 3)$

The main result of this section is the following.

THEOREM 1.

$$c_1 n^{4/3} \log\log n/\log n < U_2(n; 3) < c_2 n^{4/3}.$$

*Proof.* We first prove the upper bound. Let $G_1$ and $G_2$ be two 3-graphs, each with $n$ vertices and $e$ edges. We will successively remove isomorphic subgraphs $H$ from the $G_i$, thereby decreasing the number $e$ of edges currently remaining in each of the original graphs. The subgraph $H = H(e)$ removed will depend on the current value of $e$. We distinguish two ranges for $e$.

(i) $e \geqslant n^{5/3}$. In this case we repeatedly remove a common subgraph $H(e)$ having at least $e^2/\binom{n}{3}$ edges. The existence of such an $H(e)$ is guaranteed by Lemma 1. If $e_i$ denotes the number of edges remaining in each hypergraph after $i$ such subgraphs have been removed then

$$e_{i+1} \leqslant e_i - \frac{e_i^2}{\binom{n}{3}}. \tag{7}$$

Letting $\alpha_i = e_i/\binom{n}{3}$ we have

$$\alpha_{i+1} \leqslant \alpha_i - \alpha_i^2.$$

Since $\alpha_i < 1$ and $i^{-1} - i^{-2} < (i+1)^{-1}$, it follows by induction that $\alpha_i < i^{-1}$ for all $i$. Thus, after $n^{4/3}$ steps, the remaining graphs have at most $n^{5/3}$ edges.

(ii) $e < n^{5/3}$. For this range, we repeatedly apply Lemma 3. Let $e_0$ denote the number of edges each graph has at the beginning of this process. In general, if $e_i$ denotes the number of edges remaining after $i$ applications of Lemma 3, then

$$e_{i+1} \leqslant e_i - \sqrt{\frac{e_i}{5n}}. \tag{8}$$

Setting $\alpha_i = 5e_i n$, we have $\alpha_{i+1} \leqslant \alpha_i - \sqrt{\alpha_i}$. By hypothesis, $\alpha_0 \leqslant 5n^{8/3}$. Suppose

$$\alpha_i \leqslant (\sqrt{5}\, n^{4/3} - i/2)^2$$

for some $i \geqslant 0$. Then,

$$\begin{aligned}\alpha_{i+1} &\leqslant (\sqrt{5}\, n^{4/3} - i/2)^2 - \sqrt{5}\, n^{4/3} + i/2\\ &\leqslant (\sqrt{5}\, n^{4/3} - (i+1)/2)^2.\end{aligned}$$

Therefore, after at most $2\sqrt{5}\,n^{4/3}$ steps, all edges in each graph will have been removed. Since, the total number of steps required in (i) and (ii) is at most $(2\sqrt{5}+1)\,n^{4/3}$ then we have proved

$$U_2(n;3) \leqslant c_2 n^{4/3}$$

as required.

The lower bound is obtained by proving the existence of two hypergraphs $G_1$ and $G_2$ with $cn^{5/3}$ edges with the property that any common subgraph has at most $c'n^{1/3}\log n/\log\log n$ edges.

Let $G_1$ consist of the disjoint union of $n^{2/3}$ copies of complete 3-graphs on $n^{1/3}$ vertices. We remark here that although $n^{2/3}$ and $n^{1/3}$ may not be integers, such statements are always made with the implicit understanding that the hypergraphs (and quantities) involved may have to be adjusted slightly by adding or deleting (asymptotically) trivial subgraphs (and amounts) so as to make stated inequalities true.

$G_2$ will be a 3-graph having the following properties:

(a) There is a point $v_1$ such that $v_1 \in \bar{e}$ for all $\bar{e} \in E(G_2)$;

(b) Consider the ordinary (2-) graph $G'$ with $V(G') = \{v_2, \dots, v_n\}$ and $E(G') = \{\bar{e} - \{v_1\} : \bar{e} \in E(G_2)\}$. Then $G'$ has $\binom{n^{1/3}}{3} n^{2/3}$ edges.

(c) Any induced subgraph of $G'$ on $n^{1/3}$ points has at most $n^{1/3}\log n/\log\log n$ edges.

The existence of such a $G_2$ follows from the following probability argument.

Consider the set $\mathscr{F}$ of all ordinary (2-) graphs with $n$ vertices and $e = \binom{n^{1/3}}{3} n^{2/3}$ edges. A graph $F \in \mathscr{F}$ is said to be *bad* if there exists a set of $n^{1/3}$ points such that the induced subgraph on these vertices has at least $n^{1/3}\log n/\log\log n$ edges. The number of such bad graphs $F \in \mathscr{F}$ is bounded above by

$$A = \binom{n}{n^{1/3}} \binom{n^{2/3}}{n^{1/3}\log n/\log\log n} \binom{\binom{n}{2} - n^{1/3}\log n/\log\log n}{e - n^{1/3}\log n/\log\log n}.$$

A straightforward calculation shows

$$\frac{A}{\binom{\binom{n}{2}}{e}} \leqslant \left\{ \frac{n}{n^{1/3}/e} \left( \frac{n^{2/3}}{\dfrac{n^{1/3}\log n}{e\log\log n}} \cdot \frac{n^{5/3}}{n^2} \right)^{\log n/\log\log n} \right\}^{n^{1/3}} < 1.$$

Thus,

$$A < \binom{\binom{n}{2}}{e} = |F|$$

so that *some* graph $G' \in \mathscr{F}$ is *not* bad.

Now, let us consider a common subgraph of $G_1$ and $G_2$. $H$ must be connected since all edges in $G_2$ contain the common vertex $v_1$. Also, $|V(H)| \leqslant n^{1/3}$ since any connected component of $G_1$ has at most $n^{1/3}$ vertices. Finally, property (c) of $G_2$ implies

$$|E(H)| \leqslant n^{1/3} \log n/\log\log n.$$

Since $G_1$ and $G_2$ each have at least $n^{5/3}/10$ edges then

$$U(\{G_1, G_2\}) \geqslant c_1 n^{4/3} \log\log n/\log n.$$

This completes the proof. ∎

## Bounds on $U_k(n; 3)$

In this section we consider $U$-decompositions of $k \geqslant 3$ 3-graphs. As might be expected, our bounds are not as tight as in the case $k = 2$.

THEOREM 2. *For any* $\varepsilon > 0$,

$$c_3 n^{2-2/k-\epsilon} < U_k(n; 3) < c_4 n^{2-1/k}.$$

*Proof.* Again, we first attack the upper bound. Let $G_1, G_2, ..., G_k$ be $k$ 3-graphs with $n$ vertices and $e$ edges. There are two possibilities.

(i) $e \geqslant n^{3-2/k}$. In this case repeatedly remove a common subgraph (guaranteed by Lemma 1) having at least $e^k/\binom{n}{3}^{k-1}$ edges. Let $e_i$ denote the number of edges remaining in each graph after $i$ such subgraphs have been removed. Then

$$e_{i+1} \leqslant e_i - \frac{e_i^k}{\binom{n}{3}^{k-1}}.$$

Letting $\alpha_i = e_i/\binom{n}{3}$ we obtain

$$\alpha_{i+1} \leqslant \alpha_i - \alpha_i^k.$$

Since $\alpha_i = e/\binom{n}{3} < 1$ and

$$(i)^{-1/(k-1)} - (i)^{-k/(k-1)} \leqslant (i+1)^{-1/(k-1)}$$

then it follows by induction that

$$\alpha_i \leqslant (i)^{-1/(k-1)} \qquad \text{for all } i,$$

i.e.,

$$e_i \leqslant (i)^{-1/(k-1)} \binom{n}{3}.$$

After $n^{2-1/k}$ such subgraphs have been removed, the number of edges remaining in each graph is at most

$$(n^{2-1/k})^{-1/(k-1)} \binom{n}{3} \leqslant n^{3-(2-1/k)(1/(k-1))} < n^{3-2/k}.$$

(ii) $e < n^{3-2/k}$. In this case we repeatedly apply Lemma 3. Let $e_i$ denote the number of edges each (hyper) graph has after $i$ applications of Lemma 3 (with $e_0$ denoting the initial number of edges on each graph at the beginning of this step). Thus,

$$e_{i+1} \leqslant e_i - \sqrt{\frac{e_i}{5n}}.$$

As in the proof of Theorem 1, it can be shown that this implies

$$5e_i n \leqslant (\sqrt{5}\, n^{2-1/k} - i/2)^2.$$

Therefore, after at most $2\sqrt{5}\, n^{2-1/k}$ steps all edges have been removed from all $G_i$.

Taking count of the number of subgraphs removed in each of the two ranges for $e_t$ we conclude

$$U_k(n; 3) \leqslant c_4 n^{2-1/k}$$

as required.

The lower bound on $U_k(n; 3)$ will be proved using probability arguments. More precisely, we claim that for all $\varepsilon > 0$, there exist 3-graphs $G_1, G_2, \ldots, G_k$ with $n$ vertices and $n^{3-2/k}$ edges such that any subgraph common to all of them has at most $n^{1+\epsilon}$ edges, provided $n$ is sufficiently large. Elementary counting arguments show that the number of $k$-sets of 3-graphs with $n$ vertices and $n^{3-2/k}$ edges which contain a common subgraph with at least $n^{1+\epsilon}$ edges is less than

$$B = \binom{\binom{n}{3}}{n^{1+\epsilon}} (n!)^k \binom{\binom{n}{3} - n^{1+\epsilon}}{n^{3-2/k} - n^{1+\epsilon}}^k.$$

Since

$$\frac{B}{\binom{\binom{n}{3}}{n^{3-2/k}}^k} \leqslant \frac{n^{3n^{1+\epsilon}} n^{kn} n^{(3-2/k)k \cdot n^{1+\epsilon}}}{n^{(1+\epsilon)n^{1+\epsilon}} e^{kn} n^{3kn^{1-\epsilon}}} < 1$$

for $n$ sufficiently large then

$$B < \left(\frac{\binom{n}{3}}{n^{3-2/k}}\right)^k$$

and so, there exists a $k$-set of such graphs $G_1, G_2, ..., G_k$ with any common subgraph having at most $n^{1+\epsilon}$ edges. Thus,

$$U_k(n; 3) \geqslant U(\{G_1, ..., G_k\}) \geqslant \frac{e}{n^{1+\epsilon}} \geqslant n^{2-2/k-\epsilon}$$

for $n$ sufficiently large.

This completes the proof of Theorem 2. ∎

## Bounds on $U_2(n; r)$

This section will investigate bounds for general $r$-graphs. There are two cases, depending on the parity of $r$.

THEOREM 3. *For r even,*

$$c_5 n^{r/2} < U_2(n; r) < c_6 n^{r/2}.$$

*Proof.* Let $G_1$ and $G_2$ be two $r$-graphs, each with $n$ vertices and $e$ edges. There are two possibilities.

(i) $e \geqslant n^{r/2}$. For this case we apply Lemma 1 repeatedly, removing common subgraphs having at least $e^2/\binom{n}{2}$ edges. If $e_i$ denotes the current number of edges remaining after $i$ steps, then it can be shown by methods similar to those used in Theorems 1 and 2 that $e_i \leqslant \binom{n}{r}/i$. Thus, after at most $n^{r/2}$ steps there are at most $n^{r/2}$ edges left.

(ii) $e < n^{r/2}$. In this case we simply remove one edge at a time.

Combining the two processes, the decomposition requires at most $2n^{r/2}$ and so,

$$U_2(n; r) \leqslant c_6 n^{r/2}.$$

The lower bound is established by constructing two hypergraphs $G_1$ and $G_2$ with $cn^{r/2}$ edges for which the largest common subgraph has a single edge. To begin with, let $G_1'$ be the (hyper)graph defined by $V(G_1') = \{v_1, ..., v_n\}$ and $E(G_1') = \{\{v_1, ..., v_{r/2}\} \cup \bar{e}: \bar{e} \subseteq \{v_{r/2+1}, ..., v_n\},\ |\bar{e}| = r/2\}$. $G_1$ will be formed by selecting an arbitrary set of $c_5 n^{r/2}$ edges from $G_1'$.

$G_2$ will be an $r$-graph with $\binom{n}{r}/\binom{r}{r/2}\binom{n}{r/2}$ edges having the property that any two edges of $G_2$ intersect in at most $r/2 - 1$ vertices. The existence of such a $G_2$ is guaranteed by the following considerations. Let $S$ be an arbitrary $r$-subset of $\{1, 2,..., n\}$. The number of $r$-sets which intersect $S$ in $i$ elements is $\binom{r}{i}\binom{n-r}{r-i}$. The total number of $r$-sets which intersect $S$ in more than $r/2 - 1$ elements is $\sum_{j=0}^{r/2} \binom{r}{j}\binom{n-r}{j}$. Therefore, there must exist a family $\mathscr{F}$ of $r$-sets such that:

(a) any two $r$-sets in $\mathscr{F}$ intersect in at most $r/2 - 1$ elements;

(b) $|\mathscr{F}| \geqslant \dfrac{\binom{n}{r}}{\sum_{j=0}^{r/2} \binom{r}{j}\binom{n-r}{j}} \geqslant \dfrac{\binom{n}{r}}{\binom{r}{r/2}\binom{n}{r/2}} \geqslant c_5 n^{r/2}$.

Note that any two edges in $G_1$ intersect in at least $r/2$ elements. Thus, the largest common subgraph of $G_1$ and $G_2$ has just one edge. This implies

$$U_2(n; r) \geqslant U(\{G_1, G_2\}) \geqslant c_5 n^{r/2}.$$

and the proof of Theorem 3 is complete. ∎

THEOREM 4. *For r odd,*

$$c_7 n^{(r-1)^2/(2r-3)} \frac{\log\log n}{\log n} < U_2(n; r) < c_8 n^{r/2}.$$

*Proof.* The upper bound proof follows the same lines as the corresponding result in Theorem 3.

For the lower bound, we consider the following two $r$-graphs $G_1$ and $G_2$ on $n$ vertices. $G_1$ consists of $n^{(r-1)/(2r-3)}$ disjoint copies of complete $r$-graphs on $n^{(r-2)/(2r-3)}$ vertices. Observe that $G_1$ has $c' n^{(r^2-r-1)/(2r-3)}$ edges. For $G_2$ we will take a hypergraph satisfying the following properties:

(a) There is a vertex $v_1$ which belongs to all edges of $G_2$;

(b) $G_2$ has $c' n^{(r^2-r-1)/(2r-3)}$ edges;

(c) Consider the $(r-1)$-graph $G'$ given by $V(G') = V(G) - \{v_1\}$ and $E(G') = \{\bar{e} - \{v_1\} : \bar{e} \in E(G_2)\}$. Then any induced subhypergraph of $G'$ on $n^{(r-2)/(2r-3)}$ points has at most $n^{(r-2)/(2r-3)} \log n/\log\log n$ edges.

The (probabilistic) proof that such a graph $G_2$ exists is very similar to that used in Theorem 1 and is omitted.

Any common subgraph of $G_1$ and $G_2$ must be connected and has at most $n^{(r-2)/(2r-3)}$ vertices. Thus, it has at most $n^{(r-2)/(2r-3)} \log n/\log\log n$ edges. It follows from this that

$$U_2(n; r) > c_7 n^{(r-1)^2/(2r-3)}. \quad ∎$$

## CONCLUDING REMARKS

We close this section with the final result of the paper. Its proof uses no new techniques and will not be included.

THEOREM 5. *For all $r \geqslant 3$ and all $k$,*

$$n^{r-1-r/k} \leqslant U_k(n; r) \leqslant n^{r-1-1/k},$$

*for $n$ sufficiently large.*

## REFERENCES


1. F. R. K. CHUNG, P. ERDÖS, R. L. GRAHAM, S. M. ULAM, AND F. F. YAO, Minimal decompositions of two graphs into pairwise isomorphic subgraphs, *in* "Proceedings, Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing (1979)," pp. 3–18.
2. F. R. K. CHUNG, P. ERDÖS, AND R. L. GRAHAM, Minimal decompositions of graphs into mutually isomorphic subgraphs, *Combinatorica* 1 (1981), 13–24.


Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium