SOME MORE PROBLEMS ON ELEMENTARY GEOMETRY

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In the May 1975 issue of this Gazette I published a paper on geometric problems. I here solve one of them and state a few new ones.

I stated in my previous paper the following problem: Is it true that to every k there is an n_k so that if there are given n_k points in the plane in general position (i.e. no three on a line no four on a circle) one can always find k of them so that all the $\binom{k}{3}$ triples determine circles of different radii.

I overlooked at that time that a simple and straightforward argument gives

$$n_k \leq k + 2\binom{k-1}{2}\binom{k-1}{3}. \tag{1}$$

To prove *(1)* let $x_1,\ldots,x_m$, $m = k + 2\binom{k-1}{2}\binom{k-1}{3}$ be m points in general position and let $x_1,\ldots,x_\ell$ be a maximal set of points so that all the $\binom{\ell}{3}$ triples determine cirlces of different radii. If $\ell \geq k$ there is nothing to prove. Thus assume $\ell < k$ and we will arrive at a contradiction. Denote by $r_1,\ldots,r_{\binom{\ell}{3}}$ the radii of the circles determined by $\{x_i,x_j,x_k\}$, $1 \leq i < j < k \leq \ell$. It follows from the maximality property of $x_1,\ldots,x_\ell$ that for every $u, \ell < u \leq m$ the radius of one of the circles

$$\{x_i,x_j,x_u\},\ 1 \leq i < j \leq \ell < u \leq m$$

equals one of the r's. Observe that there are at most two circles of radius r passing through two points x_i and x_j. Thus the $n - \ell$ points x_a, $\ell < u \leq m$ must lie on at most $2\binom{\ell}{2}\binom{\ell}{3}$ circles and since they are in general position each of these circles contains at most one of these points. Thus finally $(\ell < k)$

$$m \leq \ell + 2\binom{\ell}{2}\binom{\ell}{3} \leq k - 1 + 2\binom{k-1}{2}\binom{k-1}{3}$$

which proves *(1)*.

Probably *(1)* is very far from being best possible.

A few days ago in conversation with J. Hammer we asked the following question: Let $f(n)$ be the largest integer so that any set of n points in the plane - no three on a line contains at least $f(n)$ convex subsets. Determine or estimate $f(n)$. It seems unlikely that an exact formula can be found for $f(n)$. I proved that there are two constants c_1 and c_2 so that

$$n^{c_1 \log n} < f(n) < n^{c_2 \log n} \tag{2}$$

An old question of E. Klein (Mrs. Szekeres) states: Let m_k be the smallest integer so that if m_k points are given, no three on a line, then one can always select k of them which form the vertices of a convex k-gon. Szekeres and I proved

$$n^{k-2} + 1 \le m_k \le \binom{2k-4}{k-2}. \tag{3}$$

Szekeres conjectured that there is equality on the left of (3). (3) implies (2) without much trouble. The upper bound of (2) is an immediate consequence. Consider a set of n points so that no subset of more than $\left[\frac{\log n}{\log 2}\right] + 1 = t$ points is convex. The existence of such a set is guaranteed by (3). Thus clearly

$$f(n) \le \sum_{i=0}^{t} \binom{n}{i} < n^{c_2 \log n}.$$

The proof of the lower bound in (2) is a little more complicated. Let $x_1, \ldots, x_n$ be any n points in the plane no three on a line. Put $[\sqrt{n}] = T$. By (3) every subset of size T of our n points has a convex subset of size r where

$$r > \frac{\log T}{\log 4} \ge \frac{\log n}{4}.$$

Thus our set $x_1, \ldots, x_n$ contains at least $\binom{n}{T}$ convex subsets of size r but these r sets are not necessarily distinct. The same convex set $x_{i_1}, \ldots, x_{i_r}$ occurs at most $\binom{n-r}{T-r}$ times (since there are $\binom{n-r}{T-r}$ sets of size T containing $x_{i_1}, \ldots, x_{i_r}$). Thus

$$f(n) > \binom{n}{T} \Big/ \binom{n-r}{T-r} > \left(\frac{n}{T}\right)^r > n^{c_1 \log n}$$

which completes the proof of (2).

Probably there is a constant c so that

$$\lim_{n=\infty} \log f(n)/(\log n)^2 = c.$$

I then somewhat modified our original question: Let $h(n)$ be the smallest integer so that any set of n points no three of them on a line determine at least $h(n)$ convex subsets which have no x_i in their interior. I have no satisfactory bounds for $h(n)$. E. Klein's original proof easily gives that every set of $n \ge 5$ points contains a convex quadrilateral with no point in its interior. I can not prove a similar theorem for a convex pentagon and perhaps one can give for every n, n points so that every convex pentagon determined by these points contains at least one of the

n points in its interior*.

It is a curious fact in combinatorics that finite problems are often very much more difficult than infinite ones. For example I proved that every subset ζ of E_n (the n dimensional Euclidean space), $(\zeta) = m \geq \aleph_0$ contains a subset ζ_1, $(\zeta_1) = m$ so that all the distances between points of ζ_1 are distinct. The analogous finite problem is very difficult. Let m be a finite number and denote by $g_n(m)$ the largest integer so that every set $x_1,\ldots,x_m$ in E_n contains a subset of $g_n(m)$ points all whose distances are distinct. I can not even determine $g_1(m)$.

Ulam and I to call attention to this situation made the following joke: During the second world war the U.S. navy had the slogan: The difficult we do right away the impossible takes some time. We say: The infinite we do right away, the finite may take some time.

REFERENCES

[1] P. Erdős and G. Szekeres, "A combinatorial problem in geometry", *Composito Math.* 2 (1935), 463-470.

[2] P. Erdős and G. Szekeres, "On some extremal problems in elementary geometry", *Ann. Univ. Sci.* Budapest 3-4 (1961), 313-320.

[3] P. Erdős, "Some remarks on set theory", *Proc. Amer. Math. Soc.* 1 (1950), 127-141.

* Ehrenfeucht has recently proved that for sufficiently large n there is a convex pentagon which contains no points of the set in its interior.

AUSTRALIAN MATHEMATICAL SOCIETY

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As announced in the first circular this S.R.I. is being planned to permit a large amount of interaction between mathematicians in different fields. Series of expository lectures are being arranged for the morning sessions, while afternoons are given over to more specialized sections.

Distinguished visitors will include *Jacques Louis Lions* of the Collège de France who will speak on Control Theory and P.D.E.'s, and *Tosio Kato* of the University of California at Berkeley, both for the second and third weeks, and *Cathleen Synge Morawetz* of the Courant Institute for the third week.

At present 17 specialized sections are being planned. Further details can be obtained by writing to the section organisers.