PARTITIONS INTO SUMMANDS OF THE FORM $[m\alpha]$

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§1. Asymptotic estimates for the number of partitions of the integer n into summands chosen from an arithmetic progression have been derived by several authors, see for example Meinardus [3]. In this note we investigate a natural extension which has not previosusly appeared in the literature. We shall study the asymptotic behaviour of the numbers $p_\alpha(n)$ and $q_\alpha(n)$, the number of partitions of n into summands and distinct summands, respectively, chosen from the sequence $[m\alpha]$, $m = 1,2,\ldots$ where $\alpha > 1$ is an irrational number and $[x]$ denotes the largest integer $\le x$. If $\gamma = \alpha - [\alpha]$ then for almost all $\gamma \in (0,1)$ in the Lebesgue sense we shall obtain asymptotic formulae (given in Theorem 2 below) for $p_\alpha(n)$ and $q_\alpha(n)$. However, when γ is of finite class, that is, there does not exist a number γ such that as $\ell \to \infty$

$$\ell^{1+\lambda+\epsilon}|\sin \ell\gamma\pi| \to \infty \tag{1.1}$$

for every positive ϵ, we can only deduce

$$\log p_\alpha(n) = \pi\sqrt{\frac{2n}{3\alpha}} + O(n^\delta)$$

$$\log q_\alpha(n) = \pi\sqrt{\frac{2n}{3\alpha}} + O(n^\delta)$$

for every positive δ. This is closely connected with the well known fact the larger the class of γ, defined to be the largest γ satisfying equation (1.1), the less evenly distributed is $m\alpha - [m\alpha]$. It is worth noting that if $\alpha = [\alpha] + p/q + \epsilon_q$ with ϵ_q very small, then for a long stretch the sequence $[m\alpha]$ is the union of arithmetic progressions whose difference is small compared to their length.

Finally we point out that no significant difference arises if we consider partitions into the sequence $[m\alpha+\beta]$.

§2. In this section we first apply the results of Roth and Szekeres [4]. Their results hold subject to the conditions:

PROC. SEVENTH MANITOBA CONFERENCE ON
NUMERICAL MATH. AND COMPUTING, 1977, pp. 371-377.

(I) $S = \lim_{m\to\infty} \frac{\log [m\alpha]}{\log m}$ exists;

(II) $\inf \{(\log k)^{-1} \sum_{m=1}^{k} || [ma] ||^{2}\} \to \infty$ with $k \to \infty$ where $\| \theta \|$ is

the distance of θ to the nearest integer and $\frac{1}{2}u_k < a \le \frac{1}{2}$. It is clear in our case that (I) holds. Roth and Szekeres [4,p.254] show that (II) is satisfied when the S in (I) is $< \frac{3}{2}$ and there exist constants k_0 and c such that, for all integers k, q satisfying $k > k_0$ and $1 < q \le 18[k\alpha]/k$, at least $cq^2 \log^2 k$ of the numbers $[\alpha],[2\alpha],\ldots,[k\alpha]$ are not divisible by q. In our case, $q \le 18\alpha$ and since it is well known [2,p.307] that the sequence $[m\alpha]$ is uniformly distributed in the integers this last condition holds. Thus condition (II) also holds and from the Roth-Szekeres results we have the following theorem.

THEOREM 1.

$$q_\alpha(n) = [2\pi \sum_{m=1}^{\infty} [m\alpha]^2 e^{x[m\alpha]}(1+e^{x[m\alpha]})^{-2}]^{-\frac{1}{2}}$$

$$\exp[\sum_{m=1}^{\infty} \{\frac{x[m\alpha]}{e^{x[m\alpha]}+1} + \log (1+e^{-x[m\alpha]})\}]$$

$$[1+0(n^{-\frac{1}{2}+\delta})]$$

where δ is any constant > 0 and x determined from

$$n = \sum_{m=1}^{\infty} [m\alpha](1+e^{x[m\alpha]})^{-1} .$$

Furthermore

$$p_\alpha(n) = [2\pi \sum_{m=1}^{\infty} [m\alpha]^2 e^{y[m\alpha]}(e^{y[m\alpha]}-1)^{-2}]^{-\frac{1}{2}}$$

$$\exp[\sum_{m=1}^{\infty} \{\frac{y[m\alpha]}{e^{y[m\alpha]}-1} - \log (1-e^{-y[m\alpha]})\}]$$

$$[1+0(n^{-\frac{1}{2}+\delta})]$$

where δ is any constant > 0 and y is determined from

$$n = \sum_{m=1}^{\infty} [m\alpha](e^{y[m\alpha]}-1)^{-1} \quad .$$

It will be convenient for the proof of our final results to have the following lemma.

LEMMA. Let $\zeta_\alpha(s) = \sum_{m=1}^{\infty} \frac{1}{[m\alpha]^s}$. Let λ denote the class of $\alpha - [\alpha]$, $\zeta_\alpha(s)$ the Riemann zeta function and $\{x\} = x - [x] - \frac{1}{2}$ if $x \neq [x]$ and $\{m\} = 0$ for integral m. Then

$$\zeta_\alpha(s) = \frac{1}{\alpha^s} \zeta(s) + \frac{s}{\alpha^{s+1}} \sum_{m=1}^{\infty} \frac{\{\alpha m\}}{m^s} + \frac{s}{2\alpha^{s+1}} \zeta(s+1) + sh(s)$$

where $h(s)$ is analytic in $R(s) > -1 + \epsilon$. Furthermore the only singularity of $\zeta_\alpha(s)$ in the region $R(s) \geq -\frac{1}{\lambda+1} + \epsilon$ is a simple pole at $s = 1$ and $|\zeta_\alpha(s)| = 0(|s|^4)$ uniformly in this region as $|s| \to \infty$. Finally $\zeta_\alpha(0) = (\alpha^{-1}-1)/2$ if λ is finite.

Proof. Let S be any positive real number. Let $s = \sigma + it$. Suppose $|s| \leq S$ and $m \geq M = S^2$.

Then

$$\frac{1}{[m\alpha]} = \frac{e^{-s \log (1- \frac{\{m\alpha\}+\frac{1}{2}}{m\alpha}}}{m^s \alpha^s}$$

$$= \frac{1}{m^s\alpha^s} + s\,\frac{\{m\alpha\}+\frac{1}{2}}{(m\alpha)^{s+1}} + \frac{s^2}{s(m\alpha)^{s+2}} \quad (\{m\alpha\}+\tfrac{1}{2})^2$$

$$+ \quad 0(\frac{S^3}{m^{3+\sigma}}) + 0(\frac{S}{m^{\sigma+2}})$$

where for fixed S the θ-terms are independent of s and $m \geq M$. Hence

$$\sum_{m>M} \frac{1}{[m\alpha]^s} = \frac{1}{\alpha^s} \sum_{m>M} \frac{1}{m^s} + \frac{s}{\alpha^{s+1}} \sum_{m>M} \frac{\frac{1}{2}+\{m\alpha\}}{m^{s+1}} + sh_1(s)$$

where $h_1(s)$ is analytic, being the sum of a uniformly convergent series of analytic functions, for $Rs \geq -1 + \epsilon$ and, moreover, $|h_1(s)| = 0(|s|^3)$ uniformly in $Rs \geq -1 + \epsilon$, $|s| \leq S$. Now

$$\left| \sum_{m\leq M} \frac{1}{[m\alpha]^s} \right| \leq M(M\alpha) = 0(|s|^4)$$

uniformly in $Rs \geq -1$, $|s| \leq S$. Since S was arbitrary we have the first part of the lemma and moreover, $|h(s)| = 0(|s|^3)$ uniformly in $Rs \geq -1 + \epsilon$.

Hardy and Littlewood [1] show that $\Sigma\{\alpha m\}^{-s}$ converges for $Rs \geq \lambda(\lambda+1)^{-1} + \epsilon$. It is well known that if $\Sigma a_n n^{-s}$ converges for $s = \sigma$, σ real, then $|\Sigma a_n n^{-s}| = 0(|s|)$ uniformly in $Rs \geq \alpha + \epsilon$. Hence $\Sigma\{m\alpha\}m^{-s-1}$ is $0\{|s|\}$ uniformly in $Rs \geq -(1+\lambda)^{-1} + \epsilon$. This, with our estimates for $h(s)$, gives the second part of the lemma. Finally, since $\zeta(0) = -\frac{1}{2}$ and $\zeta(s+1) = s^{-1} + \gamma + \ldots$ we have the third part.

<u>THEOREM 2.</u> Let $\alpha - [\alpha]$ be of class $\lambda < \infty$. Then

$$q_\alpha(n) = \frac{2^{\frac{1-3\alpha}{2\alpha}}}{4\sqrt{3\alpha}} n^{-\frac{3}{4}} e^{\pi\sqrt{\frac{n}{3\alpha}}} [1 + 0(n^{\frac{1}{2(1+\lambda)} + \epsilon})].$$

$$p_\alpha(n) = \frac{1}{2n^4\sqrt{6\alpha}} e^{\pi\sqrt{\frac{2n}{3}} - (\frac{\alpha^{-1}-1}{2})\log(\pi\sqrt{6\alpha}) + \zeta_\alpha'(0)}$$

$$[1 + 0(n^{-\frac{1}{2(1+\lambda)} + \epsilon})].$$

If $\alpha - [\alpha]$ is of infinite class then

$$\log q_\alpha(n) = \sqrt{\frac{n}{3\alpha}} + O(n^\epsilon)$$

$$\log p_\alpha(n) = \sqrt{\frac{2n}{3\alpha}} + O(n^\epsilon).$$

(For almost all $\alpha - [\alpha] \in (0,1)$ in the Lebesgue sense $\lambda = 0$.)

Proof. Let us consider $q_\alpha(n)$ and suppose $\alpha-[\alpha]$ is of class λ. If we let $s = \sigma + it$ we may rewrite the sums in Theorem 1 as follows.

$$\Sigma_1 = \sum_{m=1}^{\infty} \log (1+e^{-x[m\alpha]}) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \sum_{m=1}^{\infty} e^{-x\ell[m\alpha]}$$

$$= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \frac{1}{2\pi i} \sum_{m=1}^{\infty} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\ (x\ell[m\alpha])^{-s} ds$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\ x^{-s}(1-2^{1-s})\zeta(s+1)\zeta_\alpha(s)ds. \quad (\sigma > 1) \tag{2.1}$$

$$\Sigma_2 = \frac{-d\Sigma_1}{d\alpha} = \frac{1}{2\ i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)x^{-s}(1-2^{1-s})\zeta(s)\zeta_\alpha(s-1)ds. \ (\sigma > 2) \tag{2.2}$$

$$\Sigma_3 = \frac{-d^2\Sigma_1}{dx^2} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s+1)x^{-s-1}(1-2^{1-s})\zeta(s)\zeta_\alpha(s-1)ds. \quad (\sigma > 2) \tag{2.3}$$

It is well known that

$\Gamma(\sigma+it) = (e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}})$ as $|t| \to \infty$ hence in view of Lemma 1 we may shift the contour of integration in equation (2.1) to the line $Rs = -(1+\lambda)^{-1} + \epsilon$ and upon calculating the residue (note $(1-2^{-s})\zeta(s+1)$ is analytic everywhere) we obtain

$$\Sigma_1 = x^{-1} \frac{\pi^2}{12\alpha} + \zeta_\alpha(0) \log 2 + 0(x^{\frac{1}{1+\lambda} + \epsilon}) \qquad (2.4)$$

$$n = \Sigma_2 = x^{-2} \frac{\pi^2}{12\alpha} + 0(x^{-1 + \frac{1}{1+\lambda} + \epsilon}) \qquad (2.5)$$

$$\Sigma_3 = x^{-3} \frac{\pi^2}{12\alpha} + 0(x^{-2 + \frac{1}{1+\lambda} + \epsilon}) .$$

From 2.5 we obtain

$$x = \frac{\pi}{\sqrt{12\alpha n}} (1 + 0(n^{-\frac{1}{2} - \frac{1}{2(1+\lambda)} + \epsilon}) .$$

If we now insert this expression for x into equations (2.4), (2.5) and (2.6), we obtain the first result stated for $q_\alpha(n)$ in Theorem 2. The case $\alpha - [\alpha]$ of infinite class may be treated in the same way. Since it is well known [2,p.130] that $\lambda = 0$ for almost all numbers between zero and one we have all our results for $q_\alpha(n)$.

The results for $p_\alpha(n)$ follow in basically the same way. One slight difference arises since

$$-\Sigma \log (1 - e^{-y[m\alpha]}) = \frac{1}{2\pi i} \int_{\sigma+i\infty}^{\sigma+i\infty} \Gamma(s) y^{-s} \zeta(s+1) \zeta_\alpha(s) ds. \quad \sigma > 1$$

and since $\Gamma(s) = s^{-1} - \gamma + \ldots$, $\zeta(s+1) = \frac{1}{s} + \gamma + \ldots$ the residue at $s = 0$ is

$$\left. \frac{dy^{-s}\zeta_\alpha(0)}{ds} \right|_{s=0} = \zeta_\alpha(0) \log \frac{1}{y} + \zeta_\alpha'(0) .$$

We close with the following remarks. Hardy and Littlewood [1] show that if $\lambda > 0$, then the line $Rs = \lambda/(\lambda+1)$ is a natural boundary of $\Sigma\{m\alpha\}^{-s}$. They also conjecture, and it still seems open, that $Rs = 0$ is a natural boundard if $\lambda = 0$, unless α is a quadratic irrational (in this case the series may be continued to the entire plane). The presence of a natural boundary limits the accuracy of our estimates of the transcendental sums in Theorem 1 in a way that cannot be overcome by the calculus of residues. If one considers the number $p_A(n)$ of partitions of an integer n into summands chosen from A, our method leads us to consider the Dirichlet series $f_A(s) = \sum_{a\in A} a^{-s}$. Likely the line $Rs = 1$ is a natural boundary of $f_A(s)$ for almost all A. That is, if $\gamma_A = \sum_{a\in A} 2^{-a}$ then the set of γ_A for which $Rs = 1$ is not a natural boundary is of Lebesgue measure zero.

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