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ON THE DENSITY OF λ-BOX PRODUCTS

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If X is a topological space with density $d(X) \geq 2$, then $\mathrm{cf}\,(d((X^\kappa)_{(\lambda)})) \geq \mathrm{cf}\,\lambda$, where $(X^\kappa)_{(\lambda)}$ is the λ-box product of κ copies of X. We use this observation to get lower bounds for the function $\delta(\kappa, \lambda) = d((D(2)^\kappa)_{(\lambda)})$, where $D(2)$ is the discrete space $\{0, 1\}$. It turns out that $\delta(\kappa, \lambda)$ is usually (if not always) equal to the well-known upper bound $(\log \kappa)^{<\lambda}$. We also answer a question of Comfort and Negrepontis about necessary and sufficient conditions for $\delta(\kappa^+, \lambda) \leq \kappa$.

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1. Introduction

We use the letters α, κ, λ for cardinals, and the letter ξ for ordinals. For cardinals κ and λ, we write

$\mathrm{cf}\,\kappa = \min\{\alpha : \kappa \text{ is the sum of } \alpha \text{ cardinals} < \kappa\}$;
$\log \kappa = \min\{\alpha : 2^\alpha \geq \kappa\}$;
$\kappa^+ = \min\{\alpha : \alpha > \kappa\}$;
$\kappa^{<\lambda} = \sup\{\kappa^\alpha : \alpha < \lambda\}$.

We write $\beth_0 = \aleph_0$, $\beth_{n+1} = 2^{\beth_n}$, and

$$\beth_\omega = \sum_{n<\omega} \beth_n = \aleph_0 + 2^{\aleph_0} + 2^{2^{\aleph_0}} + \cdots .$$

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If A and B are sets, then ${}^A B$ is the set of all functions from A into B, $|A|$ is the cardinality of A,

$$P(A) = \{X : X \subseteq A\}, \qquad [A]^\lambda = \{X \subseteq A : |X| = \lambda\},$$

and

$$[A]^{<\lambda} = \{X \subseteq A : |X| < \lambda\}.$$

A set $F \subseteq {}^A B$ is *λ-dense* if, for every $X \in [A]^{<\lambda}$ and every function $\varphi : X \to B$, there is an $f \in F$ such that $f(x) = \varphi(x)$ for all $x \in X$. A sequence $\langle h_\xi : \xi < \kappa \rangle$ of functions $h_\xi \in {}^E B$ is *λ-independent* (of *λ-large oscillation* in the terminology of Comfort and Negrepontis [2, 3]) if, for every $X \in [\kappa]^{<\lambda}$ and every function $\varphi : X \to B$, there is an $e \in E$ such that

$$h_\xi(e) = \varphi(\xi) \quad \text{for all} \quad \xi \in X.$$

Let κ and λ be cardinals, $\lambda \leq \kappa^+$. We define $\delta(\kappa, \lambda)$ as the minimum cardinality of a λ-dense set $F \subseteq {}^\kappa\{0, 1\}$. It is easy to see that $\delta(\kappa, \lambda)$ is also the minimum cardinality of a set E such that there is a λ-independent sequence $\langle h_\xi : \xi < \kappa \rangle$ of functions in ${}^E\{0, 1\}$. For $\lambda \leq \kappa$, we define $\Delta(\kappa, \lambda) = \delta(\kappa, \lambda^+)$. There is no loss of generality in considering the function $\Delta(\kappa, \lambda)$ instead of $\delta(\kappa, \lambda)$, since

$$\delta(\kappa, \lambda) = \sup\{\Delta(\kappa, \alpha) : \alpha < \lambda\}.$$

Let $X_i (i \in I)$ be topological spaces, and let $\omega \leq \lambda \leq |I|^+$. The *λ-box product* $(\prod_{i \in I} X_i)_{(\lambda)}$ has basic open sets of the form $\prod_{i \in I} U_i$ where U_i is an open set in X_i and $|\{i \in I : U_i \neq X_i\}| < \lambda$; this is the usual Tychonoff product when $\lambda = \omega$, and the box product when $\lambda = |I|^+$. If $X_i = X$ for all $i \in I$, we write $(X^I)_{(\lambda)}$ instead of $(\prod_{i \in I} X_i)_{(\lambda)}$. The *density* of a topological space X, denoted by $d(X)$, is the minimum cardinality of a dense subset of X. We denote by $D(2)$ the space $\{0, 1\}$ with the discrete topology. Clearly,

$$\delta(\kappa, \lambda) = d((D(2)^\kappa)_{(\lambda)}) \quad \text{for } \omega \leq \lambda \leq \kappa^+.$$

We now list some well-known properties of the function $\Delta(\kappa, \lambda)$.

1.1. Lemma. *If $\kappa' \leq \kappa$ and $\lambda' \leq \lambda$, then $\Delta(\kappa', \lambda') \leq \Delta(\kappa, \lambda)$.*

1.2. Lemma. $2^\lambda \leq \Delta(\kappa, \lambda) \leq 2^\kappa$.

1.3. Lemma. $\Delta(\kappa, 2) \geq \log \kappa$.

1.4. Lemma. *If $\kappa \geq \omega$, then $\Delta(\kappa, \lambda) \leq (\log \kappa)^\lambda$.*

1.5 Theorem. *If $\kappa \geq \omega$ and $\lambda \geq 2$, then $2^\lambda \cdot \log \kappa \leq \Delta(\kappa, \lambda) \leq (\log \kappa)^\lambda$.*

1.6. Corollary. *If $\kappa \geq \omega$ and $\lambda \neq 1$, then $\Delta(\kappa, \lambda)^\lambda = (\log \kappa)^\lambda$.*

Lemmas 1.1 and 1.2 are trivial. For Lemma 1.3, consider any 3-dense family $F \subseteq {}^\kappa\{0, 1\}$. Then $\xi \mapsto \{f \in F : f(\xi) = 0\}$ is a one-to-one mapping of κ into $P(F)$;

hence $\kappa \leq 2^{|F|}$, i.e., $|F| \geq \log \kappa$. (We could prove in a similar way that $\Delta(\kappa, \lambda) \geq \log \kappa^{\lambda}$ whenever $\kappa \geq \omega$ and $\lambda \geq 2$; but this seems pointless, since $\log \kappa^{\lambda} \leq 2^{\lambda} \cdot \log \kappa$.) Lemma 1.4 follows from a result of Engelking and Karlowicz [6, Remark 3, p. 279], which generalizes earlier results of Fichtenholz and Kantorovitch [7], Hausdorff [8], and Tarski [11]. We indicate a proof here for the convenience of the reader. Put $\alpha = \log \kappa$. For $A \subseteq \alpha$ and $B \subseteq {}^{A}\{0, 1\}$, define $f_{A,B} : {}^{\alpha}\{0, 1\} \to \{0, 1\}$ so that

$$f_{A,B}(x) = 1 \quad \text{iff } x \upharpoonright A \in B,$$

and let

$$F = \{f_{A,B} : |A| \leq \lambda \text{ and } |B| \leq \lambda\}.$$

Then F is a λ^{+}-dense subset of ${}^{\alpha_2}2$; hence

$$\Delta(\kappa, \lambda) \leq \Delta(2^{\alpha}, \lambda) \leq |F| = \alpha^{\lambda} = (\log \kappa)^{\lambda}.$$

Now consider a fixed infinite cardinal κ. By Theorem 1.5, we have $\Delta(\kappa, \lambda) = \log \kappa$ for $2 \leq \lambda < \omega$, while $\Delta(\kappa, \log \kappa) = 2^{\log \kappa}$. I.e., if we put

$$\lambda_0 = \min\{\lambda : \Delta(\kappa, \lambda) > \log \kappa\},$$

then $\omega \leq \lambda_0 \leq \log \kappa$. In Section 2 we show that in fact $\lambda_0 \leq \operatorname{cf} \log \kappa$. Moreover, assuming the so-called singular cardinals hypothesis, we show that $\Delta(\kappa, \lambda) = (\log \kappa)^{\lambda}$ whenever $\kappa \geq \omega$ and $\lambda \geq 2$. In Section 3 we give two examples which answer a question of Comfort and Negrepontis. Our results were announced in [1].

2. A generalization of König's cofinality theorem

2.1. Theorem. *Let κ and λ be infinite cardinals, $\lambda \leq \kappa^{+}$, and let X be a topological space with $d(X) \geq 2$. Then $\operatorname{cf}(d((X^{\kappa})_{(\lambda)})) \geq \operatorname{cf} \lambda$.*

Proof. Note that $\alpha = d((X^{\kappa})_{(\lambda)})$ is an infinite cardinal, so it makes sense to talk about its cofinality. Suppose that $\operatorname{cf} \alpha < \operatorname{cf} \lambda$. Choose a dense set $S \subseteq (X^{\kappa})_{(\lambda)}$ with $|S| = \alpha$; then we can write $S = \bigcup_{i \in I} S_i$, where $|I| < \operatorname{cf} \lambda$ and $|S_i| < \alpha$ for $i \in I$. Since $|I| < \operatorname{cf} \lambda \leq \lambda \leq \kappa^{+}$, we can write $\kappa = \bigcup_{i \in I} K_i$, where the K_i's are pairwise disjoint sets of cardinality κ. Since $(X^{K_i})_{(\lambda)}$ is homeomorphic to $(X^{\kappa})_{(\lambda)}$, we have $d((X^{K_i})_{(\lambda)}) = \alpha$. Let $\pi_i : X^{\kappa} \to X^{K_i}$ be the projection mapping. Since $|\pi_i[S_i]| < \alpha$, $\pi_i[S_i]$ is not dense in $(X^{K_i})_{(\lambda)}$. Hence, for each $i \in I$, we can choose nonempty open sets $U_\xi \subseteq X$ $(\xi \in K_i)$ so that

$$|\{\xi \in K_i : U_\xi \neq X\}| < \lambda \quad \text{and} \quad (\textstyle\prod_{\xi \in K_i} U_\xi) \cap \pi_i[S_i] = \emptyset.$$

But then $|\{\xi \in \kappa : U_\xi \neq X\}| < \lambda$ since $|I| < \operatorname{cf} \lambda$. Thus $\prod_{\xi \in \kappa} U_\xi$ is a nonempty open set in $(X^{\kappa})_{(\lambda)}$, and $(\prod_{\xi \in \kappa} U_\xi) \cap S = \emptyset$. This contradicts the fact that S is dense in $(X^{\kappa})_{(\lambda)}$.

2.2 Corollary. *If $\omega \leq \lambda \leq \kappa^{+}$, then $\operatorname{cf} \delta(\kappa, \lambda) \geq \operatorname{cf} \lambda$.*

Proof. Let $X = D(2)$ in Theorem 2.1.

2.3. Corollary. *If $\omega \leqslant \lambda \leqslant \kappa$, then* $\operatorname{cf} \Delta(\kappa, \lambda) > \lambda$.

Since $\Delta(\kappa, \kappa) = 2^{\kappa}$, Corollary 2.3 generalizes the theorem of J. König that $\operatorname{cf} 2^{\kappa} > \kappa$.

2.4. Corollary. *If $\kappa \geqslant \omega$, then* $\Delta(\kappa, \operatorname{cf} \log \kappa) > \log \kappa$.

Proof. $\Delta(\kappa, \operatorname{cf} \log \kappa) \geqslant \log \kappa$ by Theorem 1.5, and $\operatorname{cf} \Delta(\kappa, \operatorname{cf} \log \kappa) \geqslant \operatorname{cf} \log \kappa$ by Corollary 2.3.

The *singular cardinals hypothesis*, abbreviated SCH, is the assertion that $\kappa^{\lambda} \leqslant 2^{\lambda} \cdot \kappa^{+}$ for all infinite cardinals κ and λ. (The SCH is equivalent to the assertion that $\kappa^{\operatorname{cf} \kappa} = \kappa^{+}$ for every singular cardinal κ such that $2^{\operatorname{cf} \kappa} < \kappa$; see Sections 6 and 8 of [9].) Clearly, the SCH follows from the generalized continuum hypothesis, but is much weaker. In fact, models of set theory violating the SCH are not easy to come by; Prikry and Silver (see [9, Section 37]) and Magidor [10] have constructed such models assuming the consistency of very large (e.g., supercompact) cardinals, and Jensen has shown that some large cardinal assumption is necessary [4]. The following theorem shows that the SCH settles all questions about the function $\Delta(\kappa, \lambda)$.

2.5. Theorem. *Assume the singular cardinals hypothesis. If $\kappa \geqslant \omega$ and $\lambda \geqslant 2$, then* $\Delta(\kappa, \lambda) = (\log \kappa)^{\lambda}$.

Proof. By Corollary 1.6, it will suffice to show that $\Delta(\kappa, \lambda)^{\lambda} = \Delta(\kappa, \lambda)$. Let $\alpha = \Delta(\kappa, \lambda)$. We may assume that $\lambda \geqslant \omega$. By Corollary 2.3 we have $\operatorname{cf} \alpha > \lambda$; hence ${}^{\lambda}\alpha = \bigcup_{\xi > \alpha} {}^{\lambda}\xi$ and

$$\alpha^{\lambda} \leqslant \sum_{\xi < \alpha} |\xi|^{\lambda} \leqslant \sum_{\xi < \alpha} 2^{\lambda} \cdot |\xi|^{+} \leqslant \alpha.$$

2.6 Corollary. *Assume the singular cardinals hypothesis. For every infinite cardinal κ, we have*

$$\Delta(\kappa, \lambda) = \begin{cases} 1 & \text{if } \lambda = 0; \\ 2 & \text{if } \lambda = 1; \\ 2^{\lambda} \cdot \log \kappa & \text{if } 2 \leqslant \lambda < \operatorname{cf} \log \kappa; \\ 2^{\lambda} \cdot (\log \kappa)^{+} & \text{if } \operatorname{cf} \log \kappa \leqslant \lambda \leqslant \kappa. \end{cases}$$

2.7. Corollary. *Assume the singular cardinals hypothesis. If $\kappa \geqslant \omega$ and $\lambda \geqslant 3$, then*

$\delta(\kappa,\lambda)=(\log\kappa)^{<\lambda}$. *Hence for every infinite cardinal* κ, *we have*

$$\delta(\kappa,\lambda)=\begin{cases}\lambda & \text{if } \lambda\leq 2;\\ 2^{<\lambda}\cdot\log\kappa & \text{if } 3\leq\lambda\leq \operatorname{cf}\log\kappa;\\ 2^{<\lambda}\cdot(\log\kappa)^{+} & \text{if } \operatorname{cf}\log\kappa<\lambda\leq\kappa^{+}.\end{cases}$$

Problem. Is the SCH needed in Theorem 2.5? I.e., it is consistent with ZFC that $\Delta(\kappa,\lambda)<(\log\kappa)^{\lambda}$ for some infinite cardinals κ and λ? For example, it is consistent with ZFC that $2^{\aleph_m}<\aleph_\omega$ for all $m<\omega$, $2^{\aleph_\omega}=\aleph_{\omega+2}$, and $\Delta(\aleph_\omega,\aleph_0)=\aleph_{\omega+1}$? How does the function $\Delta(\kappa,\lambda)$ behave in Magidor's models [10]?

3. Two examples

The *Souslin number* of a topological space X, denoted by $S(X)$, is the least cardinal λ such that no collection of pairwise disjoint nonempty open subsets of X has λ elements. Comfort and Negrepontis [3, Section 3, p. 79] consider the following conditions where $\omega\leq\kappa\leq\alpha$:

(f′) $\delta(2^\alpha,\kappa)\leq\alpha$;
(g′) $\delta(\alpha^+,\kappa)\leq\alpha$;
(c′) $S((D(2)^I)_{(\kappa)})\leq\alpha^+$ for all sets I;
(a′) $2^{<\kappa}\leq\alpha$.

They remark that (f′)$\Rightarrow$(g′)$\Rightarrow$(c′)$\Leftrightarrow$(a′), but it is apparently left open whether the first two implications can be reversed. (The implication (a′)$\Rightarrow$(c′) is attributed to S. Shelah). In an earlier paper by the same authors [2, p. 284], it is stated as an open problem whether (f′), (g′), and (c′) are equivalent. The examples given below show that the conditions are not equivalent.

First we give a counterexample to (a′)$\Rightarrow$(g′). Put $\alpha=\beth_\omega$ and $\kappa=\aleph_1$. Note that $\log\alpha=\alpha$ and $\operatorname{cf}\log\alpha=\aleph_0$; hence

$$\delta(\alpha,\kappa)=\Delta(\alpha,\aleph_0)=\Delta(\alpha,\operatorname{cf}\log\alpha)>\log\alpha=\alpha$$

by Corollary 2.4. Thus we have $\delta(\alpha^+,\kappa)\geq\delta(\alpha,\kappa)>\alpha$ while $2^{<\kappa}=2^{\aleph_0}<\alpha$.

Of course, we can only give a consistent counterexample to (g′)$\Rightarrow$(f′), since (f′) and (g′) are the same condition if $2^\alpha=\alpha^+$. Let us assume, then, that $2^{\aleph_0}<\aleph_\omega<2^{\aleph_1}$ and $2^{\aleph_m}<2^{\aleph_{m+1}}$ for all $m<\omega$; this assumption is consistent with ZFC (in fact, with ZFC+SCH) by Easton's theorem [5]. Put $\alpha=\aleph_\omega$ and $\kappa=\aleph_1$. Note that $\log\alpha^+=\aleph_1$ and $\log 2^\alpha=\aleph_\omega$. Now

$$\delta(\alpha^+,\kappa)=\Delta(\alpha^+,\aleph_0)=2^{\aleph_0}<\alpha$$

by Theorem 1.5, while

$$\delta(2^\alpha,\kappa)=\Delta(2^\alpha,\aleph_0)=\Delta(2^\alpha,\operatorname{cf}\log 2^\alpha)>\log 2^\alpha=\alpha$$

by Corollary 2.4. I.e.,

$$\delta(\alpha^+,\kappa)<\alpha<\delta(2^\alpha,\kappa).$$

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