*Colloques internationaux C.N.R.S.*
N° 260 - PROBLÈMES COMBINATOIRES ET THÉORIE DES GRAPHES

# ON MULTIGRAPH EXTREMAL PROBLEMS

W. G. BROWN
McGill University, Montréal, Canada

P. ERDÖS
Mat. Kut. Int., Reáltanoda ut 13-15, Budapest, Hongrie

and M. SIMONOVITS
Eötvös Loránd University, Budapest, Hongrie

**Résumé.** — Nous considérons des multigraphes (resp. graphes orientés) ayant au plus $r$ arêtes entre deux sommets (resp. $r$ arcs reliant un sommet à un autre). Dans cet article nous donnons des résultats connus et nouveaux concernant le problème suivant : étant donné une famille de multigraphes (resp. graphes orientés) $A_1, \ldots, A_k$ du type ci-dessus, quel est le nombre maximum d'arêtes (resp. arcs) que peut avoir un multigraphe (resp. graphe orienté) du type ci-dessus ne contenant aucun $A_i$ comme sous-multigraphe (sous-graphe) partiel.

1. **Introduction.** — In this paper we shall consider multigraphs and digraphs (= directed graphs) with bounded multiplicity : an integer $r$ is fixed and we shall assume, that the considered multigraphs or digraphs have no loops, further, if $u$ and $v$ are two vertices of a multigraph $M$, they can be joined by more than one edge, however, they cannot be joined by more than $r$ edges. In case of digraphs $u$ and $v$ cannot be joined by more than $r$ edges of the same orientation.

2. **Fundamental problem.** — Given the multigraphs (digraphs) $A_1, \ldots, A_k$ and a multigraph (digraph) $H^n$ not containing submultigraphs (subdigraphs) isomorphic to any $A_i$. How large $e(H^n)$ can be, where $e(H^n)$ denotes the number of edges ?

3. **The problem of the extremal structure.** — For a given set $A_1, \ldots, A_k$ of prohibited graphs and $r$ the maximum in the fundamental problem will be denoted by $ex(n ; A_1, \ldots, A_k)$. (The dependence on $r$ usually will not be indicated.) Similarly, $EX(n ; A_1, \ldots, A_k)$ denotes the family of graphs not containing any $A_i$ and having $ex(n ; A_1, \ldots, A_k)$ edges. (In other words $EX(n ; A_1, \ldots, A_k)$ is the family of graphs attaining the maximum in our problems. They will be called extremal graphs. The word graph will be used for « multigraphs » or « digraphs ».)

It is a well known theorem of P. Turán (1941 [1]) that if $r = 1$ and $\{ A_1, \ldots, A_k \} = \{ K_p \} = \{$ a complete graph on $n$ vertices $\}$, then

$$ex(n, K_p) = \frac{1}{2}\left(1 - \frac{1}{p-1}\right)(n^2 - s^2) + \binom{s}{2}, \quad (1)$$

where $s$ is the residue of the division $n/p - 1$ : $n = t(p-1) + s$, $0 \leqslant s \leqslant p - 2$.

For $r = 1$ for arbitrary family $A_1, \ldots, A_r$ of prohibited graphs Erdös and Simonovits proved in [2] that

$$ex(n, A_1, \ldots, A_k) = \frac{1}{2}\left(1 - \frac{1}{p-1} + o(1)\right) n^2 , \quad (2)$$

if $p = : \min (\chi(A_i) : i = 1, \ldots, k)$, where $\chi(G)$ denotes the chromatic number of $G$. Thus (2) means that $ex(n ; A_1, \ldots, A_r)$ depends first of all on the minimum chromatic number of the prohibited graphs if $r = 1$.

Later Erdös and Simonovits independently proved [3, 4, 5] that not only the number of edges but the structure of extremal graphs is also very near for $A_1, \ldots, A_k$ to the structure of extremal graph in Turán's original theorem. For $K_p$ (as Turán proved) there exists only one extremal graph (for each $n$), namely, the following one :

Let us determine $n_i$ $(i = 1, \ldots, p - 1)$ so that $n_1 + \cdots + n_{p-1} = n$ and $\left| n_i - \frac{n}{d} \right| \leqslant 1$. These conditions determine the integers $n_i$ up to a permutation. Let $K(m_1, \ldots, m_d)$ denote the complete $d$-partite graph with $m_j$ vertices in its $j$th class and let

$$T^{n,p-1} =: K(n_1, \ldots, n_{p-1})$$

with the integers defined above. According to Turán's theorem $T^{n,p-1}$ is the only extremal graph for $K_p$. Erdös and Simonovits proved among other results, that for any family $A_1, \ldots, A_k$ of prohibited subgraphs and $p = \min(\chi(A_i) : i = 1, \ldots, k)$ if $S^n$ is an extremal graph for $A_1, \ldots, A_k$, then one can change $o(n^2)$ edges in $S^n (n \to \infty)$, so that the obtained graph is just $T^{n,p-1}$.

Does anything like this hold for multigraphs (oriented graphs) too ? The answer is partly yes and partly no. This paper will try to explain the situation, in this way it will be a very brief survey, further it will give some of other newest results, but it will not contain proofs because they would be too long to be published here.

4. **The connection between the directed and multigraph problem.** — Let $\mathcal{A}$ be a family of prohibited multigraphs for a given even integer $r = 2s$. We define for the multiplicity bound $s$ the family of oriented (prohibited) graphs by taking each multigraph of $\mathcal{A}$ and orienting it in all the possible (permissible) ways. Let $\mathfrak{B}$ be the family of digraphs obtained in this way. Both $\mathcal{A}$ and $\mathfrak{B}$ are allowed to be infinite. One can easily see that :

(*a*) If $Q^n$ is an oriented graph and $S^n$ is obtained from $Q^n$ by omitting the orientation, then $Q^n$ will contain a $B_i \in \mathfrak{B}$ iff $S^n$ contains some $A_j \in \mathcal{A}$. (Here the digraph $Q^n$ is an « $s$-digraph », $S^n$ is an « $r$-multigraph ».)

(*b*) $Q^n$ is extremal for $\mathfrak{B}$ iff $S^n$ is extremal for $\mathcal{A}$. This shows that the extremal digraph problems are more general than the multigraph extremal problems. (One can ask, whether the digraph problems are really more general, than the multigraph problems. In some sense they are : if $A$ is a path of 3 vertices and two edges of multiplicity 1 and $B$ is a directed path of 3 vertices, $s = 1$, $r = 2$, then $ex(n, A) = 0(n)$, trivially, but $ex(n, B) = n^2/4$ since (1) if we orient all the edges of a $T^{n,2}$ from the first class towards the second one, the obtained graph will not contain $B$, but (2) if $n > 4$ and $e(Q^n) > [n^2/4]$, then either $Q^n$ contains a directed triangle and a $B$ in it or a pair of vertices $(u, v)$ joined in both direction. In the second case, if $Q^n$ contains no $B$, then $u$ and $v$ cannot be joined to any other vertex, thus one can easily prove $ex(n, B) = [n^2/4]$. Thus the digraph problem is more general, indeed.)

5. **The structure of extremal graphs in the general case, $r = 2$, $s = 1$.** — Most of our results concern the case $r = 2$ for multigraphs or the equivalent digraph problem for $s = 1$. Therefore we shall restrict ourselves to these two cases, and for the sake of brevity here we shall consider only the digraph problem. Our theorems below can easily be translated into the corresponding multigraph theorems.

6. **The matrix graphs.** — The most important features of $T^{n,d}$ are that the vertices are divided into a bounded number of classes and two vertices are joined depending only on whether they belong to the same classes or not, further, that the number of vertices in any class is a given proportion of the total number of vertices : $\approx n/d$. To generalize $T^{n,d}$ let us consider a $dxd$ matrix $A = \{ a_{i,j} \}$ where $a_{i,i} = 0$ or $1$, $a_{i,j} = 0$ or $2$ if $i \neq j$. If $x_1 + \cdots + x_d = n$, let $A((\underline{x}))$ be defined as follows $(\underline{x} = (x_1, \ldots, x_d))$ : $C_i$ is a class of $x_i$ vertices $(i = 1, \ldots, d)$ and a vertex $u \in C_i$ is joined to a vertex $v \in C_j (i \neq j)$ by an edge oriented from $u$ to $v$ iff $a_{i,j} = 2$. If $a_{i,i} = 1$, then let us enumerate the vertices of $C_i$ by $1, 2, \ldots, x_i$ and join any two vertices by an edge oriented from the smaller label to the greater one : put a complete acyclic graph into $C_i$.

**Theorem A** [6]. — *For any finite or infinite family of prohibited digraphs* (*for* $s = 1$) *there exists a matrix $A$ with the following properties* :

(i) *$A$ is a matrix described in the definition of matrix graphs* : $a_{i,i} = 0$ *or* $1$, $a_{i,j} = 0$ *or* $2$ *if* $i \neq j$.

(ii) *Let us consider for a fixed n all the matrix graphs* $A((x_1, \ldots, x_d))$ *with* $x_1 + \cdots + x_d$ *if $d$ is the size of $A$. Let $A(n)$ be one of them having maximum number of edges. Then $A(n)$ is a sequence of asymptotically extremal graphs* : *$A(n)$ does not contain any* $A_j \in \mathcal{A}$, *but if a digraph $Q^n$ has at least* $(1 + \varepsilon)\, e(A(n))$ *edges and* $n > n_0(\varepsilon)$, *then $Q^n$ contains at least one* $A_j \in \mathcal{A}$.

(iii) *If* $A(n) = A((x_1, \ldots, x_d))$, *then for every $i$ $x_i/n$ tends to a fixed $u_i$ depending only on $i$. This $u_i$ is positive and is the unique solution of the equations*

*for* $\underline{u} = (u_1, \ldots, u_d)$ *and* $e = (1, \ldots, 1)$

$$(\underline{u}\,; \underline{e}) = 1 \quad \textit{and} \quad (A + A^*)\,\underline{u} = 2\, g(A)\, \underline{e}\,,$$

*where*

$$g(A) = \max\left(\underline{v} A \underline{v} : (\underline{v}\,; \underline{e}) = 1\right). \tag{9}$$

Clearly, (ii) is the most important point in theorem A. Its meaning is that an almost extremal graph can be constructed in a very simple way : the complicatedness of its structure does not depend on $n$. The assertion (iii) needs some explanation : If $A$ is a matrix for which (i) holds, then

$$2\, e(A((x))) = \underline{x} A \underline{x} + 0(n). \tag{10}$$

From this it follows easily that

$$ex(n\,; \mathcal{A})/n^2 \to \tfrac{1}{2}\, g(A)\,, \tag{11}$$

(if (ii) is already known). (iii) asserts that the distribution of vertices in the almost extremal digraph is asymptotically uniquely defined if the matrix $A$ is appropriately defined.

Another theorem of [6] asserts that theorem A is the best possible :

**Theorems B.** — *Let $A = \{ a_{i,j} \}$ be a $d \times d$ matrix satisfying* (i) *of theorem A and $g(A)$ be defined by* (9). *If there exists only one solution of* (8), *then there exists a finite family $\mathcal{A}$ of prohibited digraphs for which theorem A holds with just this matrix A.*

Theorem B reflects that the situation is fairly complicated : without giving a precise explanation of the following statement we remark, that the condition that the system (9) and (8) has only one solution is not a too strong one, in some sense it expresses strict convexity of a quadratic form.

7. **Cases, where the extremal graphs have simple structure.** — There are two direct generalizations of the graph $T^{n,d}$ playing an important role in multigraph extremal or digraph extremal problems. $U^{n,d}$ be the multigraph obtained from $T^{n,d}$ by doubling each edge and $V^{n,d}$ be the multigraph obtained from $T^{n,d}$ by increasing the multiplicity of each edge by one : the edges of $T^{n,d}$ are changed into double edges and the independent pairs of vertices are joined by single edges. Let $\mathbf{U}^{n,d}$ denote the only permitted orientation (for $s = 1$) of $U^{n,d}$ and $\mathbf{V}^{n,d}$ denote the only permitted orientation of $V^{n,d}$ where the single edges are oriented acyclicaly : in other words, we put $d$ transitive tournaments into the $d$ classes of $\mathbf{U}^{n,d}$. In very many extremal graph problems (for multigraphs or digraphs) these graphs are the extremal graphs or the asymptotically extremal graphs. The first work investigating multigraph extremal problems and digraph extremal problems is due to W. G. Brown and F. Harary [7].

[7] contains many results in which the extremal graphs are always $U^{n,d}$, $V^{n,d}$, $\mathbf{U}^{n,d}$ or $\mathbf{V}^{n,d}$. Here we mention only one of them, which will be needed later.

**Theorem C.** — *Let $D'$ and $D''$ be two tournaments, at least one of which is different from the directed cycle on 3 vertices and let D be obtained from $D'$ and $D''$ by joining each vertex of $D'$ to each vertex of $D''$ by two edges of the opposite directions. If D has r vertices and $r \geqslant 5$, then*

$$ex(n, D) = 2\, e(T^{n,r+1})$$

*and $U^{n,r-1}$ is the only extremal graph for D. (If $D'$ and $D''$ has $r'$ and $r''$ vertices respectively, then $r = r' + r''$. that is, we assume that $D'$ and $D''$ have no common vertices.)*

A trivial consequence of this theorem is theorem C′ :

**Theorem C′.** — *If M is a multigraph in which any two vertices are joined by at least one edge and the chromatic number of $M^+$ consisting of the double edges of M is 2, then*

$$ex(n, M) = 2\, e(T^{n,r-1}),$$

*where $r \geqslant 7$ is the number of vertices in M.*

(Theorems $C$ and $C'$ are formulated above not in their most general forms.)

8. **New results.** — After the long introduction above the following theorems do not really need much explanation. Let us fix $r = 2$ for maximum multiplicity.

**Theorem 1.** — *Let M be a multigraph of $t + 1$ vertices for which any pair of vertices is joined by at least one edge. If $M^+$ is the graph on the vertices of M whose edges are the double edges of M and $M^+$ is q-chromatic, then none of $U^{n,t}$ and $V^{n,q}$ contains M, further one of the sequences (having the greater number of edges) is asymptotically extremal :*

*If $c > 0$ is fixed and $n > n_0(c)$, then for any multigraph $H^n$ with*

$$e(H^n) > \max\,(e(U^{n,t}), e(V^{n,q})) + cn^2$$

*$H^n$ must contain M as a submultigraph.*

**Corollary 1.** — *If M satisfies the conditions of theorem* 1 *and $h = \max\,(t, 2q)$, then*

$$ex(n\,;M) = \left(1 - \frac{1}{h} + o(1)\right) n^2 .$$

The cases $t = 2\,q$, $2\,q + 1$ are in some sense exceptional ones, therefore we shall not investigate them here, however in the other cases we formulate two theorems generalizing or sharpening theorem 1.

**Theorem 2.** — *Let M be a multigraph on $t + 1$ vertices and $M^+$ be the graph defined in theorem* 1. *If $M^+$ is $q + 1$-chromatic and $t < 2\,q$, then $\{ V^{n,q} \}$ is a sequence of asymptotically extremal graph :*

(i) *$M \not\subset V^{n,q}$ and*
(ii) *$ex(n\,;M) = e(V^{n,q}) + o(n^2)$.*

*(even if we do not assume that any pair of vertices is joined by at least one edge).*

**Theorem 3.** — *Let M be a multigraph on $t + 1$ vertices for which the graph $M^+$ defined in theorem* 1 *is $q + 1$-chromatic. If $t \geqslant 2\,q + 2$, then there exists an $n_0 = n_0(M)$ such that for $n > n_0$ $U^{n,t}$ is the only extremal multigraph for M.*

*Remark.* — The difference between the character of theorem 2 and theorem 3 is not so surprising as it may seem. The essential difference between the conditions of these theorems is, that in theorem 3 there exists an edge $e$ in $M$ such that $M - e \not\subset U^{n,t}$, while this is not necessarily so in theorem 2. However, if there exists an edge $e$ in $M$ in theorem 2 for which $M - e \not\subset V^{n,q}$, then there exists an $n_0$ such that for $n > n_0$ $V^{n,q}$ is the only extremal graph in theorem 2.

*Remark.* — Many extremal graph theorems are proved in a preprint of ours, [8], most of which gives the asymptotical value of $ex(n, \mathcal{A})$ for multigraph extremal problems, $r = 2$. In most of these results the asymptotically extremal graphs are different from $U^{n,d}$ and $V^{n,d}$.

9. **An example, where the extremal structure is complicated.** — Let $L$ be the following multigraph : the double edges form a paths $(x_1\, x_2\, x_3\, x_4\, x_5\, x_6)$ and $(x_1\, x_5\, x_3)$, $(x_4\, x_6\, x_2)$ are two triangles consisting of single edges. It can be proved that if $n$ is sufficiently large, then $ex(n\,;L)$ is asymptotically $0.7\, n^2$ and there exist only finitely many extremal graphs each having the following structure : the vertices are divided into 3 classes $C_1$, $C_2$ and $D$ and

$$|\,C_i\,| = \frac{n}{5} + 0(1)\,, \qquad |\,D\,| = \frac{3\,n}{5} + 0(1)$$

and the vertices of $D$ form a complete graph of single edges, the vertices of $C_i$ are independent, $C_1$ and $C_2$ are joined completely by single edges and $C_i$ and $D$ are joined by double edges (completely).

10. **Open questions.** — Here we mention just two open questions, the most important ones.

(i) Does theorem A generalize for $r = 3$ ? (For $r \geqslant 4$ ?).

(ii) Is the set of numbers $\lim_{n\to\infty} n^{-2}\, ex(n\,;\mathcal{A})$ or $\lim_{n\to\infty} n^{-2}\, ex(n\,;\mathcal{B})$ a well ordered set under the usual ordering of the real numbers ? (The affirmative answer would prove that an algorithm for solving digraph or multigraph extremal problems really works.) We conjecture that the answer is YES.

## References


[1] P. Turan, Egy grafelméleti szélsöértékfeladatrol, *Mat.-Fiz. Lapok* **48** (1941) 436-452.

[2] P. Erdös, M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hung.* **1** (1966) 51-57.

[3] P. Erdös, Some recent results on extremal problems in graph theory (Results), *Proc. of Theory of Graphs International Symp.*, Rome (1966) 118-123.

[4] P. Erdös, On some new inequalities concerning extremal properties of graphs, in : *Theory of Graphs, Proc. Colloq. Tihany.* (1966) 77-81.

[5] M. Simonovits, A method for solving extremal problems in graph theory. Stability problems, in : *Theory of Graphs, Proc. Colloq. Tihany.* (1966) 279-319.

[6] W. G. Brown, P. Erdos and M. Simonovits, Extremal problems for directed graphs, *Journal of Combinatorial Theory* **15** (1973) 77-93.

[7] W. G. Brown and P. Harary, Extremal Digraphs, *Combinatorial Theory and its applications*, edited by P. Erdös, A. Rényi, and V. T. Sós, North-Holland Publishing Company, Vol. I (1970) 135-198.

[8] W. G. Brown, P. Erdos· and M. Simonovits, Multigraph extremal problems I, II, Preprint (1975).