On Cycle–Complete Graph Ramsey Numbers

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ABSTRACT

A new upper bound is given for the cycle–complete graph Ramsey number $r(C_m, K_n)$, the smallest order for a graph which forces it to contain either a cycle of order m or a set of n independent vertices. Then, another cycle–complete graph Ramsey number is studied, namely $r(\leq C_m, K_n)$ the smallest order for a graph which forces it to contain either a cycle of order l for some l satisfying $3 \leq l \leq m$ or a set of n independent vertices. We obtain the exact value of $r(\leq C_m, K_n)$ for all $m > n$ and an upper bound which applies when m is large in comparison with $\log n$.

1. INTRODUCTION

The *Ramsey number* $r(C_m, K_n)$ is the smallest positive integer p such that every graph of order p contains either a cycle of order m or a set of n independent vertices. The study of $r(C_m, K_n)$ was initiated by Bondy and Erdös in [3]. Among their several results concerning Ramsey numbers for cycles, there is a proof that, for all values of m and n,

$$r(C_m, K_n) \leq mn^2. \tag{1.1}$$

In the first part of this paper we shall give an improvement of the

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Bondy–Erdös bound by proving that, for all $m \geq 3$ and $n \geq 2$,

$$r(C_m, K_n) \leq \{(m-2)(n^{1/k}+2)+1\}(n-1), \tag{1.2}$$

where $k = [(m-1)/2]$. For the particular case of $m = 4$, we shall give a further modest improvement of (1.2) by showing that

$$r(C_4, K_n) < c(n \log\log n/\log n)^2 \qquad (n \to \infty). \tag{1.3}$$

The Ramsey number $r(\leq C_m, K_n)$ is the smallest positive integer p such that every graph of order p contains either a cycle of order l for some l satisfying $3 \leq l \leq m$ or a set of n independent vertices. In one of the earliest applications of the probabilistic method in graph theory, one of the authors [P. E.] obtained a lower bound for $r(\leq C_m, K_n)$. Using a theorem of Lovász, Spencer has obtained an improved lower bound for $r(\leq C_m, K_n)$; in [10], Spencer proves that if m is fixed and n is sufficiently large, then

$$r(\leq C_m, K_n) \geq c(n/\log n)^{(m-1)/(m-2)}. \tag{1.4}$$

In this paper, we shall give the exact value of $r(\leq C_m, K_n)$ for all $m > n$ and an upper bound which applies when m is large in comparison with $\log n$. Interest in $r(\leq C_m, K_n)$ stems from several sources. In particular, recent work has pointed to the fact that the class of Ramsey numbers typified by $r(\leq C_m, K_n)$ occur very naturally in the study of Ramsey theory for multiple colors [6].

2. NOTATION

For the most part, our notation will be in conformity with that used in [1], [2], or [9]. All graphs considered will be finite, undirected, and without loops or multiple edges. The graph with vertex set V and edge set E will be denoted $G(V, E)$. The *order* of the graph is $|V|$ and its *size* is $|E|$.

For $X \subseteq V$, the subgraph of G induced by X will be denoted $\langle X \rangle$. The set of all vertices adjacent to at least one vertex of X will be denoted $\Gamma(X)$. In the special case where X consists of a single vertex, i.e., $X = \{v\}$, $\Gamma(v)$ is called the *neighborhood* of v. If u and v are two vertices of the graph, the *distance* $d(u, v)$ is the length of the shortest path which connects u and v. On occasion, in writing $\langle X \rangle$, $\Gamma(X)$, or $d(u, v)$ there will be a reason for emphasizing the identity of the graph to which these symbols refer. Accordingly, we shall write, when necessary, $\langle X \rangle_G$, $\Gamma_G(X)$, or $d_G(u, v)$.

Whenever x represents a real number, the symbols $[x]$ and $\{x\}$ will signify the greatest integer $\leq x$ and the least integer $\geq x$, respectively.

3. AN UPPER BOUND FOR $r(C_m, K_n)$

In the proof of our upper bound for $r(C_m, K_n)$, the graphical property now defined plays a central role.

Definition. Let l be a natural number. A graph G has property Π_l if, for every independent set X, $|\Gamma(X)| \geq l|X|$.

For our purposes, it will suffice to know the existence of an induced subgraph having property Π_l.

Lemma. Let $G(V, E)$ be a graph of order at least $(l+1)(n-1)$ which contains no set of n independent vertices. Then G contains an induced subgraph $\langle W \rangle$ which has property Π_l.

Proof. Assume, to the contrary, that none of the induced subgraphs of G has property Π_l. Thus, if $\langle W \rangle$ is any induced subgraph of G, there exists an independent set $X \subseteq W$ such that $Y = \Gamma_{\langle W \rangle}(X)$ satisfies $|Y| < l\,|X|$. With this property in mind, define $G_1 = G$, $W_1 = V$, and for $i = 1, 2, \ldots,$ set $W_{i+1} = W_i - Z_i$ and $G_{i+1} = \langle W_{i+1} \rangle$, where $Z_i = X_i \cup Y_i$, X_i is an independent set, and $Y_i = \Gamma_{G_i}(X_i)$ satisfies $|Y_i| < l\,|X_i|$. Since G is finite and $|X_i| \geq 1$ for $i = 1, 2, \ldots,$ there exists a positive integer M such that $W_{M+1} = \varnothing$,

$$V = \bigcup_{i=1}^{M} Z_i$$

is a partition of V, and

$$X = \bigcup_{i=1}^{M} X_i$$

is an independent set in $G(V, E)$. Since $|Z_i| < (l+1)\,|X_i|$ for $i = 1, 2, \ldots, M$, we find that $|V| < (l+1)\,|X|$ and this result contradicts the hypothesis that G is of order at least $(l+1)(n-1)$ and that it contains no set of n independent vertices. ∎

We are now prepared to prove the main result.

Theorem 1. For all $m \geq 3$ and $n \geq 2$, the cycle–complete graph Ramsey number $r(C_m, K_n)$ satisfies

$$r(C_m, K_n) \leq \{(m-2)(n^{1/k} + 2) + 1\}(n-1),$$

where $k = [(m-1)/2]$.

Proof. Assume $G(V, E)$ to be a graph of order $(l+1)(n-1)$ which contains no cycle of order m and no set of n independent vertices. We shall show that if $l \geq \{(m-2)(n^{1/k}+2)\}$, these assumptions about G lead to a contradiction.

By means of the preceding lemma, we know that G contains an induced subgraph $H = \langle W \rangle$ which has property Π_l. By heredity, H contains no C_m and no set of n independent vertices. Henceforth, we shall disregard the original graph G and, instead, focus our attention on the graph H and its assumed properties.

Let x be an arbitrary vertex of H. We may assume that H is connected. Otherwise, we would simply work within the connected component of H which contains x. Set $k = [(m-1)/2]$ and, for $i = 1, 2, \ldots, k$, define $A_i = \{v \mid d_H(x, v) = i\}$. We shall refer to the set A_i as the ith level.

A central part of our argument is the claim that for each i, $i = 1, 2, \ldots, k$, the induced subgraph $\langle A_i \rangle$ contains an independent set of at least $\{|A_i|/(m-2)\}$ vertices. The justification of this claim is based on the construction of a spanning tree, T, and the introduction of a total ordering for each of the sets A_i, $i = 1, 2, \ldots, k$. These processes are carried forth simultaneously according to a recursive procedure which we now describe. First, order the vertices of A_1 in an arbitrary way. Assuming that the process has been carried out to the ith level, proceed as follows. Make each vertex in A_{i+1} adjacent in T to the least element of A_i to which it is adjacent in H. Then order the vertices of A_{i+1} in conformity with the following requirement. If vertices y and z in A_{i+1} are adjacent in T to vertices u and v, respectively, in A_i and if $u < v$, then $y < z$.

A sequence of vertices $v_1, v_2, \ldots, v_M$ in A_i satisfying $v_1 < v_2 < \cdots < v_M$ will be called a *monotonic sequence*.

If, for such a sequence of vertices, $(v_1, v_2, \ldots, v_M)$ is a path $\langle A_i \rangle$, then $P = (v_1, v_2, \ldots, v_M)$ will be called a *monotonic path*. We now claim that since H contains no C_m, there can be no monotonic path of order $m-1$. Suppose that there were such a path, $P = (v_1, v_2, \ldots, v_{m-1})$. Let

$$d^* = \max_j d_T(v_j, v_{j+1}) = d_T(v_s, v_{s+1}).$$

A consideration of the relationship between the construction of T and the ordering of the sets A_i, $i = 1, 2, \ldots, k$, shows that, in fact, $d_T(v_r, v_t) = d^*$ for all $r \leq s$ and $t \geq s+1$. Moreover, it is apparent that, whatever the value of d^*, there exist vertices v_r and v_t such that the subpath of P, $(v_r, v_{r+1}, \ldots, v_t)$, together with the path connecting v_r and v_t in T, forms a cycle of order m. Since H contains no such cycle, we have proved that $\langle A_i \rangle$ contains no monotonic path of order $m-1$.

We now employ the pigeonhole principle to prove that $\langle A_i \rangle$ contains an independent set of at least $\{|A_i|/(m-2)\}$ vertices. To each vertex v in $\langle A_i \rangle$ assign as a label the order of the longest monotonic path in $\langle A_i \rangle$ which has v as its least element and note that, by definition, two vertices having the same label must be independent. Since there is no monotonic path of order $m-1$, the possible labels are the integers 1 through $m-2$. An application of the pigeonhole principle yields at least $\{|A_i|/(m-2)\}$ vertices having the same label and these are necessarily independent.

For $i = 1, 2, \ldots, k$, let B_i denote a maximal independent subset of A_i and let $r_i = |B_i|/|B_{i-1}|$ with $|B_0| = 1$. Since H has property Π_l, we know that $|\Gamma(B_i)| \geq l\,|B_i|$ for $i = 1, 2, \ldots, k$. Also, since $\Gamma(B_i) \subseteq A_{i-1} \cup A_i \cup A_{i+1}$ and $|B_i| \geq \{|A_i|/(m-2)\}$, it follows that for $i = 1, 2, \ldots, k$,

$$(m-2)(|B_{i-1}| + |B_i| + |B_{i+1}|) \geq l\,|B_i|. \tag{3.1}$$

In terms of the ratio, r_i, this inequality becomes

$$r_{i+1} \geq \left(\frac{l}{m-2} - 1\right) - \frac{1}{r_i}, \quad i = 1, 2, \ldots, k-1. \tag{3.2}$$

If we now set $l = \{(m-2)(n^{1/k} + 2)\}$, then

$$r_{i+1} \geq n^{1/k} + 1 - 1/r_i, \quad i = 1, 2, \ldots, k-1. \tag{3.3}$$

Since $r_1 \geq \{l/(m-2)\} > n^{1/k}$, it follows by induction using (3.3) that $r_i > n^{1/k}$ for $i = 1, 2, \ldots, k$, and hence $|B_k| = r_1 r_2 \cdots r_k > n$, contradicting our assumption that H contains no set of n independent vertices. ∎

We note that *for the case where m is even*, an improvement of (1.1) is already available from a result of Bondy and Simonovits [4], used in conjunction with Turán's theorem. With $m = 2l$, the upper bound obtained this way is, asymptotically, $(200\, ln)^{l/(l-1)}$ (l fixed, $n \to \infty$). The upper bound given by Theorem 1 is, asymptotically, $2(l-1)n^{l/(l-1)}$.

4. THE SPECIAL CASE OF $r(C_4, K_n)$

With two exceptions, the bound given by Theorem 1 represents progress toward understanding the behavior of $r(C_m, K_n)$ when m is fixed and n is large. The first exception, $m = 3$, is classical. Concerning this well studied case, it is known [cf. 7, Chap. 5] that there exist constants c_1 and c_2 such that, for all sufficiently large n,

$$\frac{c_1 n^2}{(\log n)^2} < r(C_3, K_n) < \frac{c_2 n^2(\log \log n)}{\log n}.$$

Concerning the second exception, $m = 4$, less is known. However, by making use of the method of Graver and Yackel [8], [cf. 7, pp. 26–29], one can prove a stronger statement than that which is contained in Theorem 1. The theorem which follows was first obtained by Spencer and one of the authors [P. E.], but the proof has not been published. It is included here for the sake of completeness.

Theorem 2

$$r(C_4, K_n) < c\left(\frac{n \log\log n}{\log n}\right)^2 \qquad (n \to \infty).$$

Proof. Let $G(V, E)$ be a graph of order $r(C_4, K_{n+1}) - 1$ which contains no C_4 and no set of $n+1$ independent vertices. Since the graph obtained from G by adding an isolated vertex must contain a set of $n+1$ independent vertices, we know that G contains a set S of n independent vertices. Let $T = V - S$ and, for every $X \subseteq T$, let $R(X) = \Gamma(X) \cap S$. For $k = 0, 1, \ldots, n$, define $T_k = \{x \mid x \in T, |R(x)| = k\}$, and let $N_k = |T_k|$.

Since S is not part of a larger independent set, it follows that $N_0 = 0$. Also, $N_1 \le 2n$, as we can see by the following argument. If $N_1 \ge 2n+1$, there are three vertices in T which are adjacent to the same single vertex in S. If any two of these three vertices are independent, then G has a set of $n+1$ independent vertices. Otherwise, G certainly contains a C_4.

Note that no two vertices in T can be adjacent to a common pair of vertices in S, for then G would contain a C_4. Since every vertex in $\bigcup_{k=m}^{n} T_k$ accounts for at least $\binom{m}{2}$ pairs of vertices in S, it follows that for all $m \ge 2$,

$$\sum_{k=m}^{n} N_k \le \frac{\binom{n}{2}}{\binom{m}{2}} = \frac{n(n-1)}{m(m-1)}. \tag{4.1}$$

Thus, with the choice of m left at our discretion, we may write

$$r(C_4, K_n) < r(C_4, K_{n+1}) \le 1 + 3n + \sum_{k=2}^{m} N_k + \frac{n(n-1)}{m(m+1)}. \tag{4.2}$$

The required bound on $\sum_{k=2}^{m} N_k$ can be realized by proving that if N_k is too large, then there must exist a set $A \subseteq S$ and an independent set C in $\langle T_k \rangle$ such that $R(C) \subseteq A$ and $|C| > |A|$. If this were so, then G would contain a set of at least $n+1$ independent vertices. The situation just described is illustrated in Fig. 1. The existence of such an independent set in $\langle T_k \rangle$ is tied to constraints on the edges of $\langle T_k \rangle$ dictated by the fact that

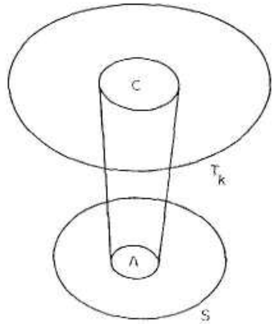

FIGURE 1. Existence of a larger independent set.

G contains a C_4. Note that if x and y are any two vertices of T, then $|R(x) \cap R(y)|$ is either 0 or 1, for, otherwise, G contains a C_4. Accordingly, we classify each edge $\{x, y\}$ in $\langle T_k \rangle$ as either *type* 0 or *type* 1. Let $M_{k,0}$ and $M_{k,1}$ denote the number of type 0 and type 1 edges, respectively, in $\langle T_k \rangle$ and let $M_k = M_{k,0} + M_{k,1}$.

Let x be an arbitrary vertex in T_k and suppose that x is incident in $\langle T_k \rangle$ with edges $\{x, y_1\}, \ldots, \{x, y_l\}$. Since G contains no C_4, the sets $R(y_i)$, $i = 1, \ldots, l$, are disjoint and, therefore, $kl \le n$. Similarly, suppose that of the incident edges, $\{x, y_1\}, \ldots, \{x, y_m\}$ are of type 1. Again, since G contains no C_4, the vertices $R(x) \cap R(y_i)$, $i = 1, \ldots, m$, are distinct and, therefore, $m \le k$. Finally, the degree bounds, $l \le n/k$ and $m \le k$, imply the edge bounds,

$$M_k \le N_k(n/2k) \tag{4.3}$$

and

$$M_{k,1} \le N_k(k/2), \tag{4.4}$$

respectively.

At this juncture, we employ the probabilistic method to prove that, unless $N_k < 5n^2/kn^{1/k}$, there exist $A \subseteq S$ and $C \subseteq T_k$ such that C is an independent set, $R(C) \subseteq A$, and $|C| > |A|$. Let Ω denote the sample space consisting of all subsets of S and, with the value of p to be chosen later, assign the probability $P(A) = p^{|A|}(1-p)^{n-|A|}$ to each $A \subseteq S$. Equivalently, each vertex in S has independent probability p of belonging to A. Corresponding to each $A \subseteq S$, define

$$B = \{x \mid x \in T_k, R(x) \subseteq A\}$$

and let C denote a maximal independent subset of B.

Let us introduce the random variables $X_A = |A|$ and $X_C = |C|$. The expected value of X_A is

$$E(X_A) = np. \tag{4.5}$$

It is difficult to ascertain the value of $|C|$, but we may obtain a lower bound for $|C|$ by subtracting the size of $\langle B \rangle$ from $|B|$. It follows that the expected value of X_C satisfies

$$E(X_C) \geq N_k p^k - (M_{k,0} p^{2k} + M_{k,1} p^{2k-1}). \tag{4.6}$$

Using the bounds given in (4.3) and (4.4), together with the fact that $p^{2k-1} - p^{2k} \geq 0$, we find that

$$E(X_C) \geq N_k \left(p^k - \frac{np^{2k}}{2k} - \frac{k(p^{2k-1} - p^{2k})}{2} \right). \tag{4.7}$$

If $p \geq k^2/(n+k^2)$, then $k(p^{2k-1} - p^{2k})/2 \leq np^{2k}/2k$ and so, by placing this restriction on p, we may be sure that

$$E(X_C) \geq N_k (p^k - np^{2k}/k). \tag{4.8}$$

Now, let us set $p = (k/2n)^{1/k}$. An elementary calculation shows that $(k/2n)^{1/k} \geq k^2/(n+k^2)$ for all $k \geq 2$ and $n \geq 1$, with equality iff $k=2$ and $n=4$. Hence, our choice of $p = (k/2n)^{1/k}$ is consistent with the previously made restriction.

If $E(X_C) > E(X_A)$, then it would be certain that G contains a set of at least $n+1$ independent vertices. As this must not be the case, we know that

$$kN_k/4n \leq n(k/2n)^{1/k}, \tag{4.9}$$

and hence

$$N_k \leq (4n^2/k)(k/2n)^{1/k} < 5n^2/kn^{1/k}, \tag{4.10}$$

where, in the last inequality, we have used the simple fact that, for all $k \geq 1$, $(k/2)^{1/k} < 5/4$. For fixed n, $kn^{1/k}$ decreases with increasing k as long as $k < \log n$. Hence, if $m < \log n$, it is certainly true that

$$\sum_{k=2}^{m} N_k < 5n^2/n^{1/m}.$$

Referring to (4.2), we have

$$r(C_4, K_n) < 1 + 3n + 5n^2/n^{1/m} + n^2/m^2.$$

Finally, by taking $m \sim \log n/(2 \log \log n)$, we obtain the bound

$$r(C_4, K_n) < c(n \log \log n/\log n)^2 \qquad (n \to \infty),$$

as claimed. ∎

5. EXACT RESULTS FOR $r(\leq C_m, K_n)$

Bondy and Erdös [3] have proved that if $m \geq n^2 - 2$, then $r(C_m, K_n) = (m-1)(n-1)+1$. In other words, if m is sufficiently large in comparison with n, then the canonical example of $n-1$ disjoint copies of K_{m-1} is critical. A similar state of affairs exists in the case of $r(\leq C_m, K_n)$. Here too, if m is sufficiently large in comparison with n, simple examples can be cited and subsequently proved to be critical. Another feature of $r(\leq C_m, K_n)$ in this realm is that, over specified intervals, it is constant, independent of m.

Theorem 3. For all $n \geq 2$,

$$r(\leq C_m, K_n) = 2n-1 \quad \text{if} \quad m \geq 2n-1,$$

and

$$r(\leq C_m, K_n) = 2n \quad \text{if} \quad n < m < 2n-1.$$

Proof. The example of $n-1$ disjoint copies of K_2 shows that, for all m, $r(\leq C_m, K_n) \geq 2n-1$. Let $G(V, E)$ be a graph of order $2n-1$ and assume that G contains no C_l for $l \leq 2n-1$. Then G is a forest and it contains a set of $\{|V|/2\} = n$ independent vertices. Thus, we have shown that $r(\leq C_m, K_n) = 2n-1$ if $m \geq 2n-1$.

The example of C_{2n-1} shows that, for all $m < 2n-1$, $r(\leq C_m, K_n) \geq 2n$. To show that $r(\leq C_m, K_n) = 2n$ if $n < m < 2n-1$, let $G(V, E)$ be a graph of order $2n$ which contains no C_l for $l \leq n+1$. We wish to show that G contains a set of n independent vertices. If G is a forest, the result is immediate. Consequently, we assume that G contains a cycle. Note that G must be a planar graph. If G were nonplanar, it would contain a subgraph homeomorphic from K_5 or $K_{3,3}$. A simple count shows that a graph which is homeomorphic from K_5 and which contains no cycle C_l for $l \leq n+1$ is of order at least $\{(10n+5)/3\}$. Similarly, a graph which is homeomorphic from $K_{3,3}$ and which contains no C_l for $l \leq n+1$ is of order at least $\{(9n+6)/4\}$. In both cases, there is a clear contradiction of the fact that G is of order $2n$.

Let X be the set of all vertices of G which lie on at least one cycle. We may assume $\langle X \rangle$ to be a 2-connected plane graph and we note that for this graph the boundary of every region is a cycle. Suppose that $\langle X \rangle$ has r regions and that it is of order p and size q. We shall show that $r < 4$. For $i = 1, \ldots, r$, let L_i denote the length of the cycle forming the boundary of the ith region. Then, since each cycle is of length at least $n+2$,

$$2q = \sum_{i=1}^{r} L_i \geq r(n+2), \tag{5.1}$$

and, by Euler's formula,

$$p = q - r + 2 \geq \frac{r}{2}(n+2) - r + 2 = \frac{r}{2}n + 2. \tag{5.2}$$

If $r \geq 4$, then $p \geq 2n+2$, in contradiction of the fact that G is of order $2n$. Our conclusion is that r must be either 2 or 3, i.e., $\langle X \rangle$ is either a cycle or a theta graph. In either case, there is a vertex, x, which belongs to every cycle of G. Hence, $G - x$ is a tree of order $2n-1$ and so it contains a set of n independent vertices. ∎

6. AN UPPER BOUND FOR $R(\leq C_m, K_n)$ WHEN m IS LARGE IN COMPARISON WITH log n

The slowly varying nature of $r(\leq C_m, K_n)$ as revealed by Theorem 3 prompts further inquiry in the form of the following question. How large must m be in order to make $r(\leq C_m, K_n) \doteq 2n$? In answer to this question, we shall show the existence of a constant A_ε such that $r(\leq C_m, K_n) \leq \{(2+\varepsilon)n\}$ whenever $m \geq [A_\varepsilon \log n]$. At the crux of our proof is the following result.

Lemma. Let δ be a fixed real number satisfying $0 < \delta < 1/2$ and let $n \geq 3$. If $G(V, E)$ is a graph of order n and size at least $\{(1+\delta)n\}$, then G contains a cycle C_l for some l satisfying $3 \leq l \leq 2[\log n/\log(1+\delta)]$.

Proof. For the case of $n = 3$, $\{(1+\delta)3\} \geq 4$ and the lemma holds vacuously. For the case of $n = 4$, $\{(1+\delta)4\} \geq 5$ and $2[\log 4/\log(1+\delta)] \geq 6$. A graph of order 4 and size at least 5 contains a C_3 and so the stated proposition certainly holds. We now take $n > 4$ and assume that the proposition holds for every m satisfying $3 \leq m < n$.

Let x be an arbitrary vertex of G and define $A_0 = \{x\}$ and $A_i = \{v \mid d(x, v) = i\}$ for $i = 1, 2, \ldots$. Set $k = [\log n/\log(1+\delta)]$ and define

$$A = \bigcup_{i=1}^{k} A_i.$$

We now assume that, contrary to the stated proposition, G contains no cycle of order $l \leq 2k$. It follows that $\langle A \rangle$ is a tree.

We may assume that for $j = 0, 1, \ldots, k$,

$$\sum_{i=1}^{j+1} |A_i| > (1+\delta) \sum_{i=0}^{j} |A_i|; \tag{6.1}$$

otherwise, since $\langle A \rangle$ is a tree, the graph $G\text{-}X$ where

$$X = \bigcup_{i=0}^{j} A_i,$$

is a graph of order $m < n$ and size at least $\{(1+\delta)m\}$. In this case, G would contain a cycle of order $l \le 2[\log m/\log(1+\delta)]$, contrary to our assumption.

From inequality (6.1) we obtain

$$|A_{j+1}| > (1+\delta) + \delta \sum_{i=1}^{j} |A_i|, \quad j = 0, 1, \ldots, k. \tag{6.2}$$

By induction, it follows that

$$|A_j| > (1+\delta)^j, \qquad j = 1, 2, \ldots, k+1. \tag{6.3}$$

In particular, since $k+1 > \log n/\log(1+\delta)$, our assumption that G contains no cycle of length $l \le 2k$ has led to the absurd conclusion that $|A_{k+1}| > n$. ∎

We are now prepared to prove the previously stated upper bound.

Theorem 4. Let $\varepsilon > 0$ be fixed. There exists a corresponding constant A_ε such that

$$r(\le C_m, K_n) \le \{(2+\varepsilon)n\}$$

whenever $m \ge [A_\varepsilon \log n]$.

Proof. Let us set $\delta = \varepsilon/2(2+\varepsilon)$ and $A_\varepsilon = 2/\log(1+\delta)$. Let $G(V, E)$ be a graph of order $p = \{(2+\varepsilon)n\}$ and let $H_1, \ldots, H_k$ denote the connected components of G. If, for some component H, $|E(H)| \ge (1+\delta)|V(H)|$, our lemma shows that H, and hence G, contains a cycle C_l for some l satisfying $3 \le l \le [A_\varepsilon \log n]$. If not, i.e., if $|E(H)| < (1+\delta) \mid V(H)$ for every component, then by deleting at most $\{\delta p\}$ appropriately chosen edges, we obtain a forest F of order p. Now we know that F contains a set of at least $\{p/2\}$ independent vertices. Upon reinstatement of the deleted edges, G is still in possession of a set of at least $\{p/2\} - \{\delta p\} = n$ independent vertices. ∎

7. QUESTIONS

Our present understanding of the behavior of $r(C_m, K_n)$ and $r(\le C_m, K_n)$ still leaves much to be desired. This is perhaps most apparent in the case of m fixed and n large, where we lack asymptotic formulas for either

$r(C_m, K_n)$ or $r(\leq C_m, K_n)$. At present, we only know that

$$c_1(n/\log n)^{(m-1)/(m-2)} < r(\leq C_m, K_n) \leq r(C_m, K_n) < c_2 n^{1+1/[(m-1)/2]}.$$

A second problem area concerns the behavior of $r(C_m, K_n)$ as a function of m, when n is fixed. From [3], we know that if $m \geq n^2 - 2$, then

$$r(C_m, K_n) = (m-1)(n-1)+1,$$

and so, eventually, the Ramsey number increases montonically with m. We now pose two questions:

(i) What is the smallest value of m such that $r(C_m, K_n) = (m-1)(n-1)+1$? It is conjectured that this formula holds for all $m \geq n$.

(ii) What value of m gives the minimum value of $r(C_m, K_n)$? From the bounds quoted above, we know that if n is fixed, but suitably large, then

$$r(C_m, K_n) > r(C_{2m-1}, K_n) \quad \text{and} \quad r(C_m, K_n) > r(C_{2m}, K_n)$$

for sufficiently small values of m. It is possible at that for a suitably large fixed value of n, $r(C_m, K_n)$ first decreases monotonically, then attains a unique minimum, then increases monotonically with m.

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