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On chagnes of signs in infinite series

I. BOROSH, C. K. CHUI, and P. ERDŐS

1. Introduction and results

The main purpose of this paper is to prove the following results.

Theorem 1. *Let $\{a_n\}$ be a sequence of positive real numbers monotonically decreasing to zero such that $\sum a_n=\infty$. Then there exist signs $\varepsilon(n)=\pm 1, n=1, 2, \ldots$, such that for every integer $m\geqq 1$ and every integer b with $0\leqq b\leqq m-1$*

$$\sum_{n\equiv b\ (\mathrm{mod}\ m)} \varepsilon(n)a_n = 0.$$

The above theorem will be a trivial consequence of the following result which is the main theorem in this paper.

Theorem 2. *Let $\{a_n\}$ be a sequence as in Theorem 1 and let $s_{n,j}$, $n=1, 2, \ldots$ and $j=0, \ldots, n!-1$ be real numbers satisfying the conditions*

$$\sum_{\substack{j=0 \\ j\equiv d\ (\mathrm{mod}\,(n-1)!)}}^{n!-1} s_{n,j} = s_{n-1,d} \tag{1.1}$$

for $n=2, 3, \ldots$ and $0\leqq d\leqq (n-1)!-1$. Then there exist signs $\varepsilon(n)=\pm 1$, $n=1, 2, \ldots$, such that

$$\sum_{\substack{k=1 \\ k\equiv j\ (\mathrm{mod}\,n!)}}^{\infty} \varepsilon(k)a_k = s_{n,j} \tag{1.2}$$

for $n=1, 2, \ldots$ and $0\leqq j\leqq n!-1$.

When $s_{n,j}=0$ for all n and j, (1.1) follows automatically. Since every arithmetic progression with modulus m is a disjoint union of $(m-1)!$ arithmetic progressions with modulus $m!$, Theorem 1 follows from Theorem 2. Also, by using this argument and (1.2), we see that in Theorem 2 each series $\sum_{n\equiv b(\mathrm{mod}\ m)} \varepsilon(n)a_n$ is, in fact, prescribed for all b and m, $0\leqq b\leqq m-1$. It is clear that conditions (1.1) on $\{s_{n,j}\}$ are necessary.

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This work was motivated by the following known result.

Theorem A. *Let* $\sum_{n=1}^{\infty}|a_n|<\infty$ *such that*

$$A_m \equiv \sum_{n\equiv 0 \pmod m} a_n = 0$$

for all $m=1,2,\ldots$. *Then* $a_1=a_2=\ldots=0$.

This result appears as a problem in [4, p. 23] and a proof is given by Kátai [3]. We now give a very simple proof. Let $p_1, p_2, \ldots$ be the sequence of all primes in increasing order and let $P_k=p_1\ldots p_k$. Then for every k

$$0=\sum_{d|P_k}\mu(d)A_d=\sum_{d|P_k}\mu(d)\sum_{l=1}^{\infty}a_{dl}=\sum_{n=1}^{\infty}\Big(\sum_{d|(n,P_k)}\mu(d)\Big)a_n=a_1+\sum_{\substack{n=2\\(n,P_k)=1}}^{\infty}a_n.$$

Here and throughout $\mu(n)$ denotes, as usual, the Möbius function.

Since $(n,P_k)=1$, $n>1$, implies $n>p_k$, we can therefore conclude that $a_1=0$ by taking $k\to\infty$. By using a standard change of index (cf. [2]) we have $a_1=a_2=\ldots=0$.

The sequence $a_n=\mu(n)/n$ satisfies $A_m=0$ for all $m=1,2,\ldots$ (cf. [1]) and this shows that the condition $\sum|a_n|<\infty$ cannot be omitted in Theorem A. Our attempt to prove that Theorem A is sharp lead us to formulate Theorem 1, which shows that Theorem A is sharp when $\{|a_n|\}$ is monotone. We remark, however, that the monotonicity condition in Theorem 1 can be weakened. In fact, we have the following result.

Theorem 3. *Let* $\sum a_n$ *be a divergent series of nonnegative real numbers with* $a_n\to 0$. *A necessary and sufficient condition for the existence of signs* $\varepsilon(n)=\pm 1$, $n=1,2,\ldots$, *such that for all* $m\geqq 1$ *and all* b, $0\leqq b\leqq m-1$, *we have*

$$\sum_{n\equiv b \pmod m}\varepsilon(n)a_n=0,$$

is that

$$\sum_{n\equiv b \pmod m}a_n=0 \quad \text{or} \quad \infty \tag{1.3}$$

for all $m\geqq 1$ *and all* b *with* $0\leqq b\leqq m-1$.

That the condition (1.3) is necessary follows by applying Theorem A to the sequence $a_n^*=a_{nm+b}$. The proof of the other direction is along the line of the proof in Section 2. However, with the lack of monotonicity the argument is much more involved and will not be presented here.

2. Proof of the main theorem

In this section we will prove Theorem 2. Throughout this section we always assume that $\{a_n\}$ is a sequence satisfying the hypothesis of Theorem 1, namely $a_n \downarrow 0$ and $\sum a_n = \infty$.

The following lemma is well-known.

Lemma 2.1. *For every real number t, there exist signs $\varepsilon(n) = \pm 1$, $n = 1, 2, \ldots$, such that $\sum \varepsilon(n) a_n = t$.*

We now state and prove our main lemma.

Lemma 2.2. *Let $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$ and m an integer not smaller than 2 be given, and $s_0, \ldots, s_{m-1}$ be real numbers with*

$$s = s_0 + \ldots + s_{m-1}. \tag{2.1}$$

Suppose that N_1 and signs $\varepsilon(n) = \pm 1$, $n = 1, \ldots, N_1$, are chosen satisfying

$$a_{N_1} < \varepsilon/2m^3 \quad \text{and} \tag{2.2}$$

$$\left|\sum_{n=1}^{N_1} \varepsilon(n) a_n - s\right| < \varepsilon. \tag{2.3}$$

Then there exists an $N_2 > N_1$ and signs $\varepsilon(n)$, $n = N_1 + 1, \ldots, N_2$, such that

$$a_{N_2} < \varepsilon_2, \tag{2.4}$$

$$\left|\sum_{\substack{n=1 \\ n \equiv b \,(\mathrm{mod}\, m)}}^{N_2} \varepsilon(n) a_n - s_b\right| < \varepsilon_1 \tag{2.5}$$

for $0 \leqq b \leqq m-1$, and

$$\left|\sum_{n=1}^{N} \varepsilon(n) a_n - s\right| < m\varepsilon \tag{2.6}$$

for all N, $N_1 \leqq N \leqq N_2$.

Proof of Lemma 2.2. Let

$$S_b(k) = \sum_{\substack{n=1 \\ n \equiv b \,(\mathrm{mod}\, m)}}^{k} \varepsilon(n) a_n - s_b$$

and

$$S(k) = \sum_{b=0}^{m-1} S_b(k) = \sum_{n=1}^{k} \varepsilon(n) a_n - s.$$

It should be noted that in the above definitions the signs $\varepsilon(n)$ have to be specified. From (2.3) we get

$$|S(N_1)| = \left|\sum_{b=0}^{m-1} S_b(N_1)\right| < \varepsilon. \tag{2.7}$$

However, it is possible that $|S_b(N_1)|$ is large for some values of b. We will define the signs $\varepsilon(n)$ in two steps. First we will find N^* and signs $\varepsilon(n)$ for $n=N_1+1, \dots, N^*$ such that

$$|S_b(N^*)| < \varepsilon \quad \text{and} \quad |S(N)| < m\varepsilon \tag{2.8}$$

for $0 \leqq b \leqq m-1$ and $N_1 \leqq N \leqq N^*$. In the second step, we define N_2 and signs $\varepsilon(n)$ for $n=N^*+1, \dots, N_2$ so that (2.4), (2.5), and (2.6) hold.

Step 1. If $|S_b(N_1)| < \varepsilon$ for $b=0, \dots, m-1$, we choose $N^*=N_1$ and (2.8) holds. Assume therefore that $|S_b(N_1)| \geqq \varepsilon$ for some values of b; say, we have u values of b with $S_b(N_1) \geqq \varepsilon$ and v values of b with $S_b(N_1) \leqq -\varepsilon$. As can be seen from the following description of the process, we can further assume, without loss of generality, that $u \leqq v$. Note that $u+v \leqq m$. It is sufficient to show that we can find $N_1^* > N_1$ and signs $\varepsilon(n)$, $n=N_1+1, \dots, N_1^*$, such that if u_1 and v_1 are defined in an analogous way to u and v when $S_b(N_1)$ is replaced by $S_b(N_1^*)$, then

$$u_1+v_1 < u+v \tag{2.9}$$

and

$$|S(N)| < 2\varepsilon \quad \text{for all } N, \quad N_1 \leqq N \leqq N_1^*. \tag{2.10}$$

We define the signs in blocks of $2vm$ terms, written for simplicity of the presentation in $2v$ rows of m terms in each row. We also assume, without loss of generality, that

$$S_b(N_1) > \varepsilon \qquad \text{for} \quad b = 0, \dots, u-1;$$

$$S_b(N_1) < -\varepsilon \quad \text{for} \quad b = u, \dots, u+v-1;$$

$$|S_b(N_1)| \leqq \varepsilon \qquad \text{for} \quad b = u+v, \dots, m.$$

The sign pattern is illustrated in the following diagram:

$$\begin{array}{r|c|c|c}
 & - - \dots - & + + \dots + & + - + - \dots \\
v+u & \vdots \quad \vdots & \vdots \quad \vdots & - + - + \dots \\
 & - - \dots - & + + \dots + & \vdots \ \vdots \ \vdots \ \vdots \\
\hline
 & - - \dots - & - - \dots - & \vdots \ \vdots \ \vdots \ \vdots \\
v-u & \vdots \quad \vdots & \vdots \quad \vdots & \\
 & - - \dots - & - - \dots - & - + - + \dots \\
 & \underbrace{\qquad\qquad}_{u} & \underbrace{\qquad\qquad}_{v} & \underbrace{\qquad\qquad}_{m-(u+v)}
\end{array}$$

When we pass one such block, the change in each $S_b(\cdot)$ is equal to the sum of the terms corresponding to column b of the diagram. After such a block, $S_b(\cdot)$ decreases for $0 \leqq b \leqq u-1$ and increases for $u \leqq b \leqq u+v-1$. For $u+v \leqq b \leqq m-1$, $S_b(\cdot)$ is alternating. We note that in each block there are vm positive signs and vm negative signs.

Since $\sum a_n = \infty$ and $\{a_n\}$ is monotone, it is clear that $\sum_{n \equiv b \pmod m} a_n = \infty$. Therefore, after a number of blocks, we eventually get a change of sign for some b^* with $0 \leqq b^* \leqq u+v-1$. Let N_1^* be the first index for which the change of sign occurs. We have $a_{N_1^*} < a_{N_1} < \varepsilon/2m^3 < \varepsilon$, and $|S_{b^*}(N_1^*)| < a_{N_1^*} < a_{N_1} < \varepsilon$ which gives $u_1+v_1 < u+v$. It remains to show that $|S(N)| < 2\varepsilon$ for $N_1 \leqq N \leqq N_1^*$. To prove this, we set $N = N_1 + 2vmq + r$ where $0 \leqq r < 2vm$ and $q \geqq 0$. Then

$$S(N) = S(N_1) + \sum_{i=0}^{q-1} \sum_{j=1}^{2vm} \varepsilon(N_1+2ivm+j)\, a_{N_1+2ivm+j} +$$

$$+ \sum_{k=1}^{r} \varepsilon(N_1+2vmq+k)\, a_{N_1+2vmq+k} \equiv S(N_1) + \Sigma_1 + \Sigma_2.$$

Write also $\Sigma_1 \equiv \sum_{i=1}^{q-1} T_i$ where T_i is the sum of all terms in the ith block. Let A_i be the sum of all terms with positive signs and B_i the negative of the sum of all terms with negative signs in the ith block. Then $T_i = A_i - B_i$. Since there are vm positive and vm negative terms in each block and the sequence $\{a_n\}$ is monotone decreasing, we clearly obtain $A_i > B_{i+1}$ and $B_i > A_{i+1}$ for each i. Hence

$$\Sigma_1 = \sum_{i=1}^{q-1} (A_i - B_i)$$

and

$$-B_0 \leqq \Sigma_1 \leqq A_0,$$

so that

$$|\Sigma_1| < vm \frac{\varepsilon}{2m^3} < \frac{\varepsilon}{2m}.$$

Also,

$$|\Sigma_2| < 2vm \frac{\varepsilon}{2m^3} \leqq \frac{\varepsilon}{m}.$$

This gives

$$|S(N)| \leqq |S(N_1)| + |\Sigma_1| + |\Sigma_2| < \varepsilon + \frac{3}{2}\frac{\varepsilon}{m} < 2\varepsilon.$$

Step 2. We apply the standard technique used in the proof of Lemma 1 to each of the divergent series $\sum_{\substack{n=N^*+1 \\ n \equiv b \pmod m}}^{\infty} a_n$, $b=0, \dots, m-1$. We can therefore pick $N_2 > N^*$ and signs $\varepsilon(n)$, $n = N^*+1, \dots, N_2$, such that (2.4) and (2.5) hold. For any N such that $N^* \leqq N \leqq N_2$, we have

$$|S(N)| \leqq \sum_{b=0}^{m-1} |S_b(N)| \leqq \sum_{b=0}^{m-1} |S_b(N^*)| \leqq m\varepsilon$$

which, combined with (2.8), gives (2.6).

This completes the proof of Lemma 2.2.

Lemma 2.3. *Under the hypotheses of Theorem 2, there exist a sequence* $N_1<N_2<\dots$ *and signs* $\varepsilon(n)=\pm 1$, $n=1,2,\dots$, *such that for each* $j=1,2,\dots$,

$$\left|\sum_{\substack{n=1\\ n\equiv l\,(\mathrm{mod}\,(j-1)!)}}^{N} \varepsilon(n)a_n - s_{j-1,l}\right| < \frac{2}{j!} \tag{2.11}$$

for all l *and* N *with* $0\leqq l\leqq (j-1)!-1$ *and* $N_{j-1}\leqq N\leqq N_j$.

Proof of Lemma 2.3. The proof will be by induction. By Lemma 1 with $t=s_{1,0}$, we can find N_1 and signs $\varepsilon(n)$, $n=1,\dots,N_1$, such that

$$a_{N_1} < \frac{1}{2\cdot 2^4 2!} \quad\text{and}\quad \left|\sum_{n=1}^{N_1}\varepsilon(n)a_n - s_{1,0}\right| < \frac{1}{2\cdot 2!}.$$

Suppose that $N_1<\dots<N_{k-1}$ and signs $\varepsilon(n)=\pm 1$, $n=1,\dots,N_{k-1}$, have been determined such that for $j=1,\dots,k-1$,

$$a_{N_j} < \frac{1}{2(j+1)^4(j+1)!}, \tag{2.12}$$

$$\left|\sum_{\substack{n=1\\ n\equiv b\,(\mathrm{mod}\,j!)}}^{N_j} \varepsilon(n)a_n - s_{j,b}\right| < \frac{1}{(j+1)(j+1)!}, \quad 0\leqq b\leqq j!-1, \tag{2.13}$$

and (2.11) hold. We will now determine $N_k>N_{k-1}$ and signs $\varepsilon(n)$, $n=N_{k-1}+1,\dots,N_k$, such that (2.12), (2.13) and (2.11) hold for $j=k$. Fix an l, $0\leqq l\leqq (k-1)!-1$, and apply Lemma 2.2 to the subsequence $\{a_n\}$, $n\equiv l\,(\mathrm{mod}\,(k-1)!)$, with $m=k$, N_1 replaced by N_{k-1}, s_b replaced by $s_{k,b}$ for b of the form $b\equiv l+v(k-1)!$, $v=0,\dots,k-1$, and

$$\varepsilon=\frac{1}{k\cdot k!},\quad \varepsilon_1=\frac{1}{2(k+1)(k+1)!},\quad \varepsilon_2=\frac{1}{2(k+1)^4(k+1)!}.$$

By the induction hypotheses (2.12) and (2.13), we see that the conditions (2.2) and (2.3) in Lemma 2.2 hold. Hence there exists an $N_{k,l}>N_{k-1}$ such that

$$a_{N_{k,l}} < \frac{1}{2(k+1)^4(k+1)!}, \tag{2.14}$$

$$\left|\sum_{\substack{n=1\\ n\equiv b\,(\mathrm{mod}\,k!)}}^{N_{k,l}} \varepsilon(n)a_n - s_{k,b}\right| < \frac{1}{2(k+1)(k+1)!} \tag{2.15}$$

for $b=l+v(k-1)!$, $v=0,\dots,k-1$, and

$$\left|\sum_{\substack{n=1\\ n\equiv l\,(\mathrm{mod}\,(k-1)!)}}^{N} \varepsilon(n)a_n - s_{k-1,l}\right| < \frac{1}{k!},\quad N_{k-1}\leqq N\leqq N_{k,l}. \tag{2.16}$$

Let $N_k=\max\{N_{k,l}: l=0, \dots, (k-1)!-1\}\equiv N_{k,l^*}$. The signs $\varepsilon(n)=\pm 1$, $N_{k-1}\leqq n\leqq N_k$ and $n\equiv l^*(\bmod (k-1)!)$, are already defined. For $n\equiv l(\bmod (k-1)!)$ with $l\neq l^*$, we define $\varepsilon(n)=\pm 1$, $N_{k,l}+1\leqq n\leqq N_k$, as follows: Blocks of k positive signs alternate with blocks of k negative signs in each of the subsequences.

From (2.14) we have (2.12) with $j=k$. To prove (2.13) with $j=k$, let b, $0\leqq b\leqq k!-1$, be fixed and set $b\equiv l(\bmod (k-1)!)$, $0\leqq l\leqq (k-1)!-1$. Write

$$\sum_{\substack{n=1\\ n\equiv b(\bmod k!)}}^{N_k} \varepsilon(n)a_n - s_{k,b} = \left(\sum_{\substack{n=1\\ n\equiv b(\bmod k!)}}^{N_{k,l}} \varepsilon(n)a_n - s_{k,b}\right) + \sum_{\substack{n=N_{k,l}+1\\ n\equiv b(\bmod k!)}}^{N_k} \varepsilon(n)a_n \equiv \Sigma_1+\Sigma_2.$$

By (2.15), $|\Sigma_1|<1/2(k+1)(k+1)!$. From the construction the terms in Σ_2 are alternating in signs. Therefore, $|\Sigma_2|\leqq 2a_{N_{k,l}}<1/(k+1)^4(k+1)!$ from (2.14). Hence $|\Sigma_1|+|\Sigma_2|<1/(k+1)(k+1)!$ which gives (2.13) with $j=k$. To prove (2.11), and hence completing the induction proof, fix N, $N_{k-1}\leqq N\leqq N_k$, and l, $0\leqq l\leqq (k-1)!-1$. Because of (2.16) we may assume that $N>N_{k,l}$. Writing

$$\sum_{\substack{n=1\\ n\equiv l(\bmod (k-1)!)}}^{N} \varepsilon(n)a_n - s_{k-1,l} = \left(\sum_{\substack{n=1\\ n\equiv l(\bmod (k-1)!)}}^{N_{k,l}} \varepsilon(n)a_n - s_{k-1,l}\right) +$$

$$+ \sum_{\substack{n=N_{k,l}+1\\ n\equiv l(\bmod (k-1)!)}}^{N} \varepsilon(n)a_n \equiv \Sigma_3+\Sigma_4,$$

we have $|\Sigma_3|<1/k!$ by (2.16) and

$$|\Sigma_4|\leqq 4ka_{N_{k,l}}\leqq \frac{4k}{2(k+1)^4(k+1)!}<\frac{1}{k!},$$

since Σ_4 consists of blocks of k positive terms alternating with blocks of k negative terms, with possible exception at the two ends. This gives (2.11) for $j=k$, completing the proof of Lemma 2.3.

The following lemma can easily be verified by induction.

Lemma 2.4. *Let the $s_{i,j}$'s satisfy the hypotheses of Theorem 2.2. Then for* $t\geqq w$, $0\leqq b\leqq w!-1$, *and* $w=1, 2, \dots$,

$$\sum_{v=0}^{(t!/w!)-1} s_{t,b+vw!} = s_{w,b}. \tag{2.17}$$

We can now complete our proof of Theorem 2. Let $N_1<N_2<\dots$ and signs $\varepsilon(n)=\pm 1$, $n=1, 2, \dots$, be determined as in Lemma 2.3. Fix w and b with $0\leqq b\leqq w!-1$, and let $N>N_w$. Let $t\geqq w$ satisfy $N_t<N\leqq N_{t+1}$. Then by (2.17), we have

$$\Sigma_N\equiv \sum_{\substack{n=1\\ n\equiv b(\bmod w!)}}^{N} \varepsilon(n)a_n - s_{w,b} = \sum_{v=0}^{(t!/w!)-1}\left(\sum_{\substack{n=1\\ n\equiv b+vw!(\bmod t!)}}^{N} \varepsilon(n)a_n - s_{t,b+vw!}\right).$$

By (2.11) we obtain

$$|\Sigma_N| \leq \sum_{r=0}^{(t!/w!)-1} \frac{2}{(t+1)!} = \frac{2t!}{w!(t+1)!} = \frac{2}{w!(t+1)}.$$

As $N\to\infty$ we have $t\to\infty$, so that $\Sigma_N\to 0$. This completes the proof of the theorem.

3. Related problems

Theorem A is equivalent to the following theorem for analytic functions (cf. [2, 3]).

Theorem B. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *with* $\Sigma |a_n|<\infty$. *Then if*

$$s_m(f) \equiv \frac{1}{m}\sum_{k=1}^{m} f(e^{i2\pi k/m}) = 0$$

for $m=1, 2, \ldots, f$ *is identically zero.*

In [1] it is shown that the function $f(z)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}z^n$ is continuous on $|z|\leq 1$ and satisfies $s_m(f)=0$ for all $m=1, 2, \ldots$. This leads to the following problem.

Problem 1. Does there exist a function $f(z)=\sum_{n=1}^{\infty} a_n z^n$, continuous on $|z|\leq 1$, with $s_m(f)=0$ for $m=1, 2, \ldots$ and such that $\Sigma |a_n|=\infty$ and $na_n\to 0$?

A more difficult question is the following.

Problem 2. Let $a_n\geq 0$ and $\Sigma a_n=\infty$. Under what conditions on the sequence $\{a_n\}$ do there exist signs $\varepsilon(n)=\pm 1$, $n=1, 2, \ldots$, such that $f(z)=\sum_{n=1}^{\infty}\varepsilon(n)a_n z^n$ is continuous on $|z|\leq 1$ and satisfies $s_m(f)=0$ for all $m=1, 2, \ldots$?

The following theorem shows that the sign construction in Theorem 2 cannot be adapted to solve the above problem.

Theorem 4. *Let* $f(z)=\sum_{n=1}^{\infty} a_n z^n$ *be such that* $\sum_{n\equiv b \pmod m} a_n=0$ *for all* b *and* m, $0\leq b<m$, $m=1, 2, \ldots$. *Then either* $f\equiv 0$ *or* f *is not continuous on* $|z|\leq 1$.

This theorem is an immediate consequence of the following result.

Theorem 5. *Let* $f(z)=\sum_{n=1}^{\infty} a_n z^n$ *converge for all* $z=e^{i2\pi t}$, t *rational. Suppose that* $\sum_{n\equiv b \pmod m} a_n$ *converges for all* b *and* m, $0\leq b\leq m-1$, $m=1, 2, \ldots$. *Then*

$f(e^{i2\pi t})=0$ *for all rational* t *if and only if* $\sum_{n\equiv b(\mathrm{mod}\ m)} a_n=0$ *for all* b *and* m, $0\leqq b\leqq \leqq m-1$, $m=1, 2, \dots$.

Proof of Theorem 5. Let $t=c/m$, $0\leqq c\leqq m-1$, $(c, m)=1$. Then $f(e^{i2\pi t})= = \sum_{n=1}^{\infty} a_n e^{i2\pi c/m}$. Hence we have

$$f(e^{i2\pi t}) = \sum_{b=0}^{m-1} e^{i2\pi cb/m} \sum_{n\equiv b(\mathrm{mod}\ m)} a_n. \tag{3.1}$$

Therefore, if $\sum_{n\equiv b(\mathrm{mod}\ m)} a_n=0$ for all b and m, $0\leqq b\leqq m-1$, $m=1, 2, \dots, f(e^{i2\pi t})=0$ for all rational numbers t. On the other hand, if $f(e^{i2\pi t})=0$ for $t=0, 1/m, \dots, (m-1)/m$, then (3.1) gives a system of m linear homogeneous equations for $s_b= = \sum_{n\equiv b(\mathrm{mod}\ m)} a_n$, $b=0, \dots, m-1$, with coefficient matrix $(e^{i2\pi cb/m})_{0\leqq c, b\leqq m-1}$ which is clearly non-singular. Hence, $s_0=\dots=s_{m-1}=0$.

Problem 3. Let there be given $\aleph_0$ infinite sets of integers $\{A_n\}$, $n=1, 2, \dots$. Assume:

$$\sum_{m\in A_n} |a_m| = \infty, \quad n = 1, 2, \dots, \quad \text{and} \quad a_n \to 0. \tag{3.2}$$

Is it true that we can find signs $\varepsilon(m)=\pm 1$ so that

$$\sum_{m\in A_n} \varepsilon(m) a_m = 0 \quad n = 1, 2, \dots ?$$

Clearly, some conditions for the A_n's are needed. For example, the A_n's are closed with respect to Boolean operations.

Problem 4. Let $\alpha>1$, $0\leqq\beta<\alpha$. Assume that for every α and β

$$\sum_{n=0}^{\infty} a_{[n\alpha+\beta]} = 0.$$

Does it follow that $a_n=0$, $n=1, 2, \dots$? Or more generally: Let $\{A_\alpha\}$ $1\leqq\alpha<\omega_1$ be a family of $\aleph_1$ infinite sequences of integers. Find non-trivial conditions so that if $\{a_n\}$ satisfies

$$\sum_{n\in A_\alpha} a_n = 0, \quad 1\leqq\alpha<\omega_1,$$

then $a_n=0$, $n=1, 2, \dots$.

Problem 5. Our main theorem gives that there is a non-trivial power series $\Sigma\, a_n z^n$ which is zero for every $z=e^{i2\pi\theta}$, θ rational. In fact, if (3.2) holds for arithmetic progressions, there exist $\varepsilon(m)=\pm 1$ so that $\sum_{m=1}^{\infty} \varepsilon(m) a_m z^m=0$ at these points. Does this remain true if the rational multiples of π are replaced by any countable set on $|z|=1$?

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О расстановках знаков в бесконечных рядах

И. БОРОШ, Ц. К. ЧУИ и П. ЭРДЕШ

Доказана теорема.

Пусть $\{a_n\}$ — *последовательность положительных чисел, монотонно стремящаяся к нулю, причем* $\sum a_n = \infty$. *Тогда найдется такая последовательность знаков* $\varepsilon(n) = \pm 1$ $(n = 1, 2, \ldots)$, *что для каждого натурального* m *и любого целого* b *с условием* $0 \leqq b \leqq m-1$ *выполнено равенство*

$$\sum_{n \equiv b \pmod m} \varepsilon(n) a_n = 0.$$

Приведены также некоторые родственные результаты и указаны нерешенные вопросы.

I. BOROSH AND C. K. CHUI
DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION
TEXAS 77 843, USA

P. ERDŐS
MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13—15
1053 BUDAPEST, HUNGARY