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ON ADDITIVE PARTITIONS OF INTEGERS

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Given a linear recurrence integer sequence $U=\{u_n\}$, $u_{n+2}=u_{n+1}+u_n$, $n\geq 1$, $u_1=1$, $u_2>u_1$, we prove that the set of positive integers can be partitioned uniquely into two disjoint subsets such that the sum of any two distinct members from any one set can never be in U. We give a graph theoretic interpretation of this result, study related problems and discuss possible generalizations.

1. Introduction

The aim of this paper is to prove that given a linear recurrence sequence $U=\{u_n\}$, $u_{n+2}=u_{n+1}+u_n$, $n\geq 1$, $u_1=1$, $u_2>u_1$, then the set of positive integers can be partitioned uniquely into two disjoint subsets such that the sum of any two distinct members from any one set can never be in U and study related problems.

We prove our main result in Section 2. In Section 3 we give a graph theoretic interpretation of this, and look at such recurrence sequences as extremal solutions to certain problems relating to the partition of the set of integers. In Section 4 we make a brief study of some special properties of partitions generated by such recursive sequences. Finally in Section 5 we mention related problems and possible generalizations of our results.

The theorem mentioned in the first paragraph has been proved simultaneously and independently, [2, 4, 6], by Evans, Silverman and Nelson for the case $u_2=2$, (Fibonacci Numbers). But we have learned that their methods are quite different. Moreover in this paper we study the same problem in a more general setting.

The explicit theorem originated by Silverman is [7]:

Theorem. *The positive integers have a unique division into two disjoint sets with the property that a positive integer is a Fibonacci Number if and only if it is not the sum of two distinct members of the same set.*

Let A_1 and A_2 be sets of positive integers such that

$$A_1 \cup A_2 = \mathbf{N} \text{ (the set of positive integers)},$$
$$A_1 \cup A_2 = \emptyset \text{ (the empty set)},$$

then

$$A_1 = \{1, 3, 6, 8, 9, 11, 14, 16, 17, 19, \ldots\}$$
$$A_2 = \{2, 4, 5, 7, 10, 12, 13, 15, 18, \ldots\}$$

are examples of the first few terms of such sets. No two distinct elements from the same set sum to a Fibonacci Number but *every* non-Fibonacci Number is the sum of two distinct elements from the same set. Every $m = 2F_s$ is uniquely representable in this way.

2. The main theorem and its proof

Definition 2.1. Consider a set of positive integers A. Denote by $\mathbf{N} = \{1, 2, 3, 4, \ldots\}$. So $A \subseteq \mathbf{N}$. We say that A *generates an additive partition of* $\mathbf{N}$ if there exists $A_1, A_2 \subseteq \mathbf{N}$ with $\mathbf{N} = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, such that for any distinct positive integers a and b with $a, b \in A_1$ or $a, b \in A_2$ we have $a + b \notin A$.

It is the aim of this section to prove the following:

Theorem 2.2. *If $U = \{u_n\}$ is a linear recurrence sequence with $u_{n+2} = u_{n+1} + u_n$, $n \geq 1$, $u_1 = 1$, $u_2 > 1$, then U generates a unique additive partition of* $\mathbf{N}$.

Proof. (Existence.) We shall give an explicit construction of two sets A_1, A_2 which generate this additive partition. First observe that it is absolutely necessary to have $u_1, u_3, u_5, \ldots$ in the same set and $u_2, u_4, u_6, \ldots$ in the other set. So let

$$\begin{aligned} A_1 &\supseteq \{u_1, u_3, u_5, u_7, \ldots, u_{2n+1}, \ldots\} \\ A_2 &\supseteq \{u_2, u_4, u_6, \ldots, u_{2n}, \ldots\}. \end{aligned} \tag{1}$$

Let $u_2 = b$, so that $U = \{1, b, b+1, 2b+1, \ldots\}$.

Case 1. $b = 2a$ (even).

Consider two consecutive positive integers c, $c+1$ where $c < a$. Now c and $c+1$ must lie in the same set, because otherwise the number d defined by $c + d = 2a = u_2$ and $c + 1 + d = 2a + 1 = u_3$ cannot lie in either set because $d \neq c$ and $d \neq c+1$. So this forces $1, 2, \ldots, a-1$, a to lie in the same set. That is,

$$A_1 = \{1, 2, \ldots, a-1, a, \ldots\}. \tag{2}$$

Now since

$$U = \{1, 2a, 2a+1, 4a+1, \ldots\}$$

we should have

$$A_2 \supseteq \{a+1, a+2, \dots, 2a-1, 2a, \dots\}. \tag{3}$$

So far we have had no trouble and have constructed the sets up to $u_2 = 2a$. Now let's go to:

Case 2. $b = 2a+1$ (odd).

As in Case 1, if $c < a$, then c and $c+1$ must lie in the same set because otherwise if d is defined by $d+c = 2a+1 = u_2$ and $d+c+1 = 2a+2 = u_3$ then $d \neq c$ or $d \neq c+1$ and hence d does not get a place. So this forces

$$A_1 = \{1, 2, 3, \dots, a-1, \dots\}.$$

Where do we put a? If $a \in A_2$ then $a+1 \in A_1$ because $2a+1 \in U$. But then $a+a+2 = 2a+2 \in U$ and $a+2+a-1 = 2a+1 \in U$ and so $a+2$ has no position. Thus we must have $a \in A_1$. So

$$A_1 = \{1, 2, \dots, a-1, a, \dots\}. \tag{4}$$

This clearly forces

$$A_2 = \{a+1, a+2, \dots, 2a-1, 2a, 2a+1, \dots\}. \tag{5}$$

So again we have determined the two sets up to u_2 without trouble.

Choose an integer $n > u_2$. We will explicitly say whether $n \in A_1$ or $n \in A_2$ by the following construction.

Construction. If $n \in U$ then the position of n has already been determined by (1). If $n \notin U$ then there exists a unique integer m such that $u_m < n < u_{m+1}$. Denote by $i = n - u_m$. Observe $i < u_{m+1} - u_m = u_{m-1} < n$. So now inductively assume that i has been assigned a position. Now if i is never half a member of U then we assign n to the same set. If $i = \frac{1}{2}u_{k-1}$ (then $m \geq k-1$), we assign n to the same set as i if $m \neq k$ and to the opposite set if $m = k$.

We claim that the two sets A_1, A_2 constructed thus give an additive partition of **N**.

We prove this by induction. That is, assume that no two distinct members from $\{1, 2, 3, \dots, n-1\}$ from the same set add up to a member of U. We show this is true for $\{1, 2, 3, \dots, n\}$. We have shown it is true for $n = u_2$. So now $n > u_2$. Moreover by the induction assumption it suffices to consider sums $a+n$ where $a < n$. There exists a unique integer m such that

$$u_m \leq n < u_{m+1}.$$

Clearly $n = u_m$ does not cause any trouble because if $a < n$ and $a+n \in U$ then $a = u_{m-1}$, but then by (1) a and n lie in opposite sets. So we have

$$u_m < n < u_{m-1}.$$

But then

$$a + n < 2n < u_{n+3}$$

so we can only have $a + n = u_{m+1}$ or $a + n = u_{m+2}$. We show both these are not possible by our construction.

Case 1. $a + n = u_{m+1}$.

As before let $i = n - u_m < n$. Observe $a + i = (u_{m+1} - n) + (n - u_m) = u_{m-1}$. Both a and i are $< n$ and so are in $\{1, 2, \ldots, n-1\}$. By our induction assumption if $a \neq i$ then a and i lie in opposite sets. But then $i = \frac{1}{2}u_m$ and so by our construction i and $n = u_m + i$ lie in the same set. So n and a lie in opposite sets.

If $a = i = \frac{1}{2}u_{m-1}$ then again by our construction $a = i$ and $n = u_m + i$ lie in opposite sets.

Case 2. $a + n = u_{m+2}$.

In Case 1 above we showed that n and $u_{m+1} - n$ lie in opposite sets. But then by our construction $u_{m+1} - n$ and $u_m + (u_{m+1} - n)$ lie in the same set unless

$$u_{m+1} - n = \tfrac{1}{2}u_{m-1} = \tfrac{1}{2}(u_{m+1} - u_m) \tag{6}$$

which means

$$2n = u_{m+2} = a + n.$$

This would force $a = n$ which violates our assumption $a < n$. So (6) does not hold and consequently $u_{m+1} - n$ and $u_m + (u_{m+1} - n) = u_{m+2} - n = a$ lie in the same set. But then by Case 1 since n and $u_{m+1} - n$ lie in opposite sets we conclude that n and a lie in opposite sets.

So we have shown that if $a < n$, then $a + n \notin U$ if a and n lie in the same set and that proves existence.

(Uniqueness). We show that there is at most one partition possible. We already observe that (1) is necessary. Now if $n \notin U$ then there exists a unique integer m such that $u_m < n < u_{m+1}$. Set $j = u_{m+1} - n$. Observe $j < u_{m+1} - u_m = u_{m-1} < n$, so that if j has been assigned a set then n must go to the opposite set. So if j has at most one position, so does n. But then at the beginning of the existence proof we showed that the numbers $j \leq u_2$ occupied unique positions, namely (2), (3), (4) and (5). So by induction there is at most one partition possible. That completes the proof of our main theorem.

We can combine the uniqueness and construction in Theorem 2.2 into the following.

Theorem 2.3. *The unique additive partition generated by U in Theorem* 2.2 *has the following property: Given $n \notin U$, $n > u_2$, pick m so that $u_m < n < u_{m+1}$. Set $i = n - u_m$. Then if $i \neq \frac{1}{2}u_{m-1}$, n and i lie in the same set. If $i = \frac{1}{2}u_{m-1}$ then n and i lie in opposite sets.*

We shall use Theorem 2.2 quite often in later sections of this paper.

3. Extremal solutions

In this section we show that the sequences U discussed in Theorem 2.2 are extremal solutions to certain problems about additive partitions of $\mathbf{N}$. We begin by giving a graph theoretic interpretation of our result. For that we need

Definition 3.1. Consider a set $A \subseteq \mathbf{N}$. Define a graph on $\mathbf{N}$ as follows: Two distinct points of $\mathbf{N}$ are joined by a line if they sum up to a member of A. We call such a graph the *additive graph generated by A.*

If the additive graph generated by A is two colorable then let A_1 be all points of $\mathbf{N}$ of one color and A_2 the other set. Clearly A_1 and A_2 generate an additive partition of $\mathbf{N}$. Conversely if A generates an additive partition of $\mathbf{N}$ with sets A_1 and A_2, then coloring all points of A_1 with one color, and points of A_2 with the other gives a two coloration of the additive graph of A.

It is well known that a graph is two colorable if and only if it is bipartite. (For these definitions about graphs see [1].) So let us record this analysis in the following:

Proposition 3.2. *A set $A \subseteq \mathbf{N}$ generates an additive partition of $\mathbf{N}$ if and only if the additive graph generated by A is two colorable, that if and only if the additive graph generated by A is bipartite.*

Consider $A = \{a_n\}$ a strictly increasing sequence of positive integers, and the simultaneous equations

$$a + b = a_n$$
$$b + c = a_m$$
$$c + a = a_l.$$

We then get

$$a = \frac{a_n - a_m + a_l}{2}, \qquad b = \frac{a_m - a_l + a_n}{2}, \qquad c = \frac{a_l - a_n + a_m}{2}. \tag{7}$$

The necessary and sufficient condition for the three solutions in (7) to be positive is that

$$\max(a_n, a_m, a_l) < a_i + a_j \tag{8}$$

where $i \neq j$ and $i, j = n, m, l$. We may assume without loss of generality that $a_n < a_m < a_l$ so that (8) is equivalent to

$$a_n + a_m > a_l. \tag{9}$$

If we want all the solutions in (7) to be integers then we need $a_n + a_m + a_l \equiv 0 (\text{mode } 2)$. So in such a case the additive graph generated by A contains a triangle

(a, b), (b, c), (c, a) and so is not bipartite. Thus by Proposition 3.2, A does not generate an additive partition of $\mathbf{N}$. So we have

Theorem 3.3. *Let $A = \{a_n\}$ be a strictly increasing sequence of positive integers such that for some indices n, m, l, we have $a_n + a_m + a_l \equiv O(\text{mod } 2)$ $(n < m < l)$. Further if a_n, a_m, a_l lie closer together than the two term recurrence sequence in Theorem 2.2, that is $a_n + a_m > a_l$, then A does not generate an additive partition of* $\mathbf{N}$.

Remark. What Theorem 3.3 essentially says is that if $\{a_n\} = A$ grows slower than a two term recurrence sequence U and contains even numbers $(a_n + a_m + a_l \equiv 0(\text{mode } 2))$ then A does not generate an additive partition of $\mathbf{N}$. For the sequence U itself the simultaneous equations have non-negative integer solutions in (7) only when $m = n + 1$, and $l = n + 2$ but one solution is zero which we do not accept.

Theorem 3.3 tells us that Theorem 2.2 won't hold with arbitrary initial conditions u_1 and u_2. For instance $u_1 = 2p + 1$, $u_2 = 1$, where $p > 0$ is an integer does not work. Similarly $u_1 = 2p$, $u_2 = 2$ does not work also.

The condition $a_n + a_m + a_l \equiv 0(\text{mod } 2)$ is absolutely essential because if all a_n were odd then

$$A_1 = \{n \in \mathbf{N} \mid n \equiv 1(\text{mod } 2)\}$$
$$A_2 = \{n \in \mathbf{N} \mid n \equiv 2(\text{mod } 2)\}$$

is an additive partition for A.

Example. Let $A = \{n^2 \mid n \in \mathbf{N}\}$. Then A satisfies the conditions of Theorem 3.3 and hence the additive graph generated by A is not two colorable. What is the chromatic number of this graph? We feel it is infinite!

If a sequence grows faster than U discussed in Theorem 2.2, it generally generates an additive partition and possibly more than one. We can however show

Theorem 3.4. *If $A = \{a_n\}$ is a strictly increasing sequence of positive integers which grows faster than a power of 2, that is, $a_{n+1} \geq 2a_n$ then A generates infinitely many additive partitions of* $\mathbf{N}$.

Proof. First of all the elements of A can go in either of two sets. Because if $n < a_m$ then $a_m < n + a_m < 2a_m \leq a_{m+1}$. So we have infinitely many choices for the positions of $\{a_n\}$. If $n \notin A$, then pick m so that $a_m < n < a_{m+1}$. Now let $j = a_{m+1} - n$. If $j < n$, assign n to the set opposite to that of j which by induction is assumed to have been assigned a position. If $j \geq n$ then n can occupy any position because if $n' < n$ then $n' + n < 2n < 2a_{m+1} < a_{m+2}$. That proves Theorem 3.4.

We will conclude this section by showing that recursive sequences U in Theorem 2.2 are an extremal solution to a certain additive partition problem. First we need

Definition 3.5. A set $A \subseteq \mathbf{N}$ is defined to saturated if A generates an additive partition of $\mathbf{N}$ and no set B that properly contains $A \cup \{1, 2\}$, and $B \subseteq \mathbf{N}$ generates an additive partition of n. In other words if any $p \in \mathbf{N} - A$ then if A_1, A_2 is an additive partition for A, we can find distinct integers $a, b \in A_1$ or $a, b \in A_2$ with $a + b = p$ $(p > 2)$.

The reason why we consider $A \cup \{1, 2\}$ in Definition 3.5 is because 1 and 2 are never the sum of distinct positive integers.

We are now in a position to prove:

Theorem 3.6. *Let $U = \{u_n\}$ be a recursive sequence as in Theorem 2.2. Let $u_2 = b > 1$. Then if b is even, U is saturated. If b is odd then $U^* = U \cup \{b-1\}$ is saturated. Moreover $b-1$ can be made a member of the sequence by setting $u_0 = b-1$, and the recurrence relation is still satisfied.*

We require

Lemma 3.7. *Consider the unique partition A_1, A_2 of* $\mathbf{N}$ *as in Theorem 2.2, generated by U. Then if $i = \frac{1}{2}u_{m-1}$, i and u_m lie in the same set.*

Proof. Observe that by (1), (2), (3), (4) and (5) this clearly holds for the first such value of i namely $i = b/2$ or $i = (b+1)/2$. Our proof will be induction on the subscript. If Lemma 3.7 is true for $i = \frac{1}{2}u_{m-1}$, then the next even member of the sequence is u_{m+2}. So consider $j = \frac{1}{2}u_{m+2}$. Now observe $u_m + i = j$, because

$$j = \frac{u_{m+2}}{2} = \frac{u_{m+1} + u_m}{2} = \frac{2u_m + u_{m-1}}{2} = u_m + i. \tag{10}$$

Now by Theorem 2.3, and (10), j and i lie in opposite sets. So j and u_m lie in opposite sets. But then since by (1), u_m and u_{m+3} lie in opposite sets, we infer that j and u_{m+2} lie in the same set and that proves the lemma by induction.

Proof of Theorem 3.6. Since the additive partition generated by U is unique, one need only look at sets A_1, A_2 defined by (1) through (5) in Theorem 2.2. One deduces immediately that one cannot add any $p < b-1$ to U and still get an additive partition.

If b is even then $b-1$ cannot be added because the three numbers $b-1$, b, $b+1$ would satisfy the conditions of Theorem 3.6 and the additive graph would contain a triangle. Or else by writing $b = 2a$, we observe in (2) that a, $a-1 \in A_1$ and $a + a - 1 = b - 1$.

If b is odd then by straightforward inspection of (4) and (5) we see that $b-1$ can be added to U without any trouble.

So we will now show that for any $p \notin U$, $p > b$, there exists $a, b \in A_1$ or $a, b \in A_2$ with $a+b=p$. Clearly for such a p, we can pick a unique m with

$$u_m < p < u_{m+1}.$$

Set $i = p - u_m$. We have several cases.

Case 1. $i \neq \frac{1}{2}u_{m-1}$, $\quad i \neq \frac{1}{2}u_{m-3}$, $\quad i \neq u_{m-3}$.

Without loss of generality assume $u_m \in A_1$. Then by (1), $u_{m+1} \in A_2$. If $i \in A_1$ then $i + u_m = p$ and $i < u_m$ because $i < u_{m+1} - u_m = u_{m-1} < u_m$ and we will be done. So we might as well assume $i \in A_2$. Then by Theorem 2.3, $u_m + i \in A_2$. But then $(u_m + i) + (u_{m-1} - i) = u_{m+1}$ and so $u_{m-1} - i \in A_1$. Observe if $i \neq \frac{1}{2}u_{m-3}$ then $u_{m-2} + i \in A_2$ and $u_{m-1} \in A_2$. But then their sum is $u_m + i = p$. Moreover since $i \neq u_{m-3}$, $u_{m-2} + i$ and u_{m-1} are distinct and $u_{m-1} + (u_{m-2} + i) = p$ is the required representation.

Case 2. $i = \frac{1}{2}u_{m-1}$.

Then by Lemma 3.7 i and u_m lie in the same set. So $u_m + i = p$ is the required representation.

Case 3. $i = \frac{1}{2}u_{m-3}$.

Again by Lemma 3.7 i and u_{m-2} lie in the same set. So by (1) i and u_m lie in the same set and $p = u_m + i$.

Case 4. $i = u_{m-3}$.

First note that $u_m + u_{m-3} = 2u_{m-1}$. So we will show that numbers of the form $2u_m$ are representable additively. So write $2u_m = u_m + i + u_m - i$. If this is never a representation then that means $u_m + i$ and $u_m - i$ lie in opposite sets always. We will get a contradiction.

(i) Let u_m be even. Then if $i = \frac{1}{2}u_m$, $u_m - i = i$ and $u_m + i$ lie in the same set by Theorem 2.3. So we get a representation. Also this is the only representation because otherwise i and $u_m - i$ lie in opposite sets and by Theorem 2.3. i and $u_m + i$ lie in the same set.

(ii) Let u_m be odd, u_{m-1} even. In this case i and $u_m - i$ lie in opposite sets always. So for an additive representation of $2u_m$ we need $u_m + i$ and i in opposite sets and by Theorem 2.3 this happens if and only if $i = \frac{1}{2}u_{m-1}$.

(iii) Let u_m be odd, u_{m+1} even. Here also i and $u_m - i$ lie in opposite sets. So again for an additive representation we need $u_m + i$ and i in opposite sets. If $i \leq u_{m-1}$ then by Theorem 2.3 this does not happen. So let $u_{m-1} < i < u_m$. Then let j be defined by $u_m + i = u_{m+1} + j$ so that $j = i - u_{m-1}$. Since u_{m-1} is odd by Theorem 2.3, i and j lie in the same set always, unless $j = \frac{1}{2}u_{m-2}$, (u_{m-2} is even). Now $j = i - u_{m-1} < u_{m-2} < \frac{1}{2}u_{m+1}$. So j and $u_{m+1} + j = u_m + i$ lie always in the same set. So $u_m + i$ and $u_m - i$ lie in the same set it is necessary and sufficient that $j = \frac{1}{2}u_{m-2}$, that is, $i = u_{m-1} + \frac{1}{2}u_{m-2}$. So we have actually proved that $2u_m$ has a unique representation. That completes the proof of Theorem 3.6.

4. Properties of the sets A_1 and A_2

In this section we discuss some special properties of the sets A_1, A_2 of the additive partition generated by U in Theorem 2.2. For instance

Theorem 4.1. *Let $u_2 = b$ and $a = [b/2]$ where $[x]$ is the largest integer $\leq x$. Then neither sets A_1 or A_2 can contain $a+2$ consecutive integers.*

Proof. The proof will be by induction and contradiction. One notices from (2), (3), (4) and (5) that this is true up to $b+1$. Now assume we have $a+2$ consecutive integers in one set, say $n, n+1, \ldots, n+a+1$.

Case 1. There exists an integer m such that

$$u_m \leq n < n+1 < \cdots < n+a+1 < u_{m+1}.$$

Consider the integers $i_j = u_{m+1} - (n+j) \leq u_{m-1} < n$. The i_j are all less than n, and hence would lie in the set opposite to these. This would be a contradiction by induction.

Case 2. There exists an integer m so that

$$u_m < n < n+1 < \cdots < n+a+1 \leq u_{m+1}.$$

If $n+a+1 = u_{m+1}$ then consider the differences $u_{m+1} - (n+j)$ for $0 \leq j < a+1$ which gives $1, 2, \ldots, a+1$ in the same set contradicting (2) and (4). If $n+a+1 < u_{m+1}$ then it is clear by Case 1.

Case 3. There exists an integer m such that

$$n < n+1 < \cdots < n+j = u_m < n+j+1 < \cdots < n+a+1.$$

In this case by Theorem 2.3, $n+j+1$ and 1 lie in the same set while $n+j-1$ and 1 lie in opposite sets. Thus such a collection of $a+2$ numbers can never lie in the same set. We are done.

We have a companion result to Theorem 4.1.

Theorem 4.2. *If α_n and β_n denote the n^{th} members of A_1 and A_2 respectively we have*

$$|\alpha_n - \beta_n| \leq a+2.$$

Proof. Consider a member $u_m \in U$ which is odd. Without loss of generality assume $u_m \in A_1$. Denote by

$$s = \sum_{\alpha \leq u_m, \alpha \in A_1} 1,$$

Let these numbers be written in ascending order $\alpha_1, \alpha_2, \ldots, \alpha_s$. So $\alpha_s = u_m$. Now

$u_m - \alpha_i$ for $i < s$ determine the set of all $\beta_j \in A_2$, $\beta_j < u_m$. So we have

$$u_m - \alpha_{s-i} = \beta_i. \tag{11}$$

The case $u_m - \alpha = \alpha$ does not arise since u_m is odd. So by (11)

$$|\alpha_n - \beta_n| = |\alpha_{s-n} - \beta_{s-n}|. \tag{12}$$

If u_m is a member of the sequence immediately after α_n and if u_m is odd, then by (12) we can connect the difference $|\alpha_n - \beta_n|$ with $|\alpha_{s-n} - \beta_{s-n}|$ and $s - n < n$. So by induction we would infer $|\alpha_n - \beta_n| \leq a + 2$. If $u_m \in A_2$ then the above argument can be modified easily.

So the only problem is when u_m is even. In this case the only trouble in the above correspondence is caused by $\frac{1}{2}u_m$ and so there can be a discrepancy of at most 2. In the beginning by observation of (2), (3), (4) and (5), $|\alpha_n - \beta_n| = a$. So we have for the first such occurence of u_m even, $|\alpha_n - \beta_n| \leq a + 2$. But then the even u_m go in alternate sets by (1) and by Lemma 3.7 the $\frac{1}{2}u_m$ also go in opposite sets alternately. So the errors cancel out. This proves Theorem 4.2.

One of the standard ways to partition integers is as follows: pick a real number $\alpha > 1$ and denote by

$$\mathbf{N}_\alpha = \{[n\alpha] \mid n \in A\}.$$

Then (see [3] or [5])

$$\mathbf{N}_\alpha \cup \mathbf{N}_\beta = \mathbf{N}, \qquad \mathbf{N}_\alpha \cap \mathbf{N}_\beta = \emptyset \tag{14}$$

if and only if α and β are irrational and

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

In our case by Theorem 4.2 we have

$$\lim_{n \to \infty} \frac{\alpha_n}{\beta_n} = 1. \tag{15}$$

So (15) implies we cannot get a partition as in (14) because then $\alpha = \beta = 2$.

One can ask several interesting questions about the sets A_1 and A_2. We conclude this section by considering one. We showed in Theorem 3.6 that U or U^* was saturated. That is for $p > b$ there always existed an additive representation in one set. It is natural to ask the following questions. Which numbers $p \notin U$, $p > b$, are represented in both sets, and which in only one? Theorem 4.3 below gives a complete answer.

Theorem 4.3. *If $p \in U$, $p \neq b - 1$, then we can find distinct integers a, b with a, $b \in A_1$ or $a, b \in A_2$ and $a + b = p$. The only integers p that are representable in such a fashion in only one of these sets are $p = 2u_m$. In such cases the representation is unique.*

The proof of Theorem 4.3 has essentially been carried out in Theorem 3.6. For instance in Case 4 there we proved the uniqueness of representation of $2u_m$. If one goes through Cases 1, 2, and 3 more carefully, one can give a complete proof of Theorem 4.3. We do not go through the details here.

5. Related problems and possible generalizations

How would we generalize Theorem 2.2? It is probably better to look at its graph theoretic form, namely Proposition 3.2. This naturally leads to the following general question: Under what conditions can we say that an additive graph generated by A is k-colorable? When will this coloration be unique?

We can also ask for a partition of integers into two sets such that the sum of k-distinct members from any one set is never in A. (We call this a k-additive partition generated by A.) Throughout this paper we studied 2-additive partitions. Just as one can put additive partitions in the language of graphs (Proposition 3.2), additive partitions will lead to k-hypergraphs. Will we get as extremal solutions to certain problems of k-additive partitions the k-term recurrence sequences? These questions are extremely difficult to answer.

In a subsequent paper, V.E. Hoggatt will discuss further interesting properties of recursive sequences and additive partitions. We conclude this paper with the following question:

If a set A is saturated, does it necessarily generate a unique additive partition of $\mathbf{N}$? If not, under what conditions does a saturated set generate a unique additive partition of $\mathbf{N}$?

References

[1] C.Berge, Graphs and Hypergraphs (North-Holland Publishing Co., Amsterdam, 1976).
[2] R.Evans, private correspondence.
[3] A.Fraenkel, Complementary systems of integers, Amer. Math. Monthly 84 (2) (1977) 114–115.
[4] H.Nelson, private correspondence.
[5] I.Niven, Diophantine approximations, in: Interscience Tracts in Pure and Applied Mathematics, No 14, (Wiley Interscience, New York, 1963).
[6] D.L. Silverman, private correspondence.
[7] D.L. Silverman, Problems and conjectures, J. Recr. Math. 9 (4) (1976–77) 208, problem 567.