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INTERSECTION PROPERTIES OF SYSTEMS OF FINITE SETS

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ABSTRACT

Let X be a finite set of cardinality n. If $L = \{l_1, \ldots, l_r\}$ is a set of non-negative integers $l_1 < l_2 < \ldots < l_r$, and k is a natural number then by an (n, L, k)-system we mean a collection of k-element subsets of X such that the intersection of any two different sets has cardinality belonging to L. We prove that if $\mathscr{A}$ is an (n, L, k)-system, $|\mathscr{A}| > cn^{r-1}$ ($c = c(k)$ is a constant depending on k) then

(i) there exists an l_1-element subset D of X such that D is contained in every member of $\mathscr{A}$,

(ii) $(l_2 - l_1) | (l_3 - l_2) | \ldots | (l_r - l_{r-1}) | (k - l_r)$,

(iii) $\prod_{i=1}^{r} \frac{n - l_i}{k - l_i} \geqslant |\mathscr{A}|$ (for $n \geqslant n_0(k)$).

Parts of the results are generalized for the following cases:

(a) we consider t-wise intersections, $t \geqslant 2$,

(b) the condition $|A| = k$ is replaced by $|A| \in K$ where K is a set of integers,

(c) the intersection condition is replaced by the following: among $q+1$ different members $A_1, \ldots, A_{q+1}$ there are always two subsets A_i, A_j such that $|A_i \cap A_j| \in L$.

We consider some related problems. An open question:

Let $L' \subset L$. Does there exist an (n, L, k)-system of maximal cardinality $(\mathscr{A})$ and an (n, L', k)-system of maximal cardinality $(\mathscr{A}')$ such that $\mathscr{A} \supset \mathscr{A}'$?

RESULTS

Throughout, lower case latin letters denote integers, capital letters stand for sets and capital script letters for families of sets.

Let $L = \{l_1, \ldots, l_r\}$, $l_1 < l_2 < \ldots < l_r$ and K be sets of integers. By an (n, L, k)-system we mean a family $\mathscr{A}$ of subsets of a set X, $|X| = n$ such that for $A_1, A_2 \in \mathscr{A}$ we have $|A_1|, |A_2| \in K$, $|A_1 \cap A_2| \in L$. If $K = \{k\}$ then the notation (n, L, k)-system is applied, too.

A family $B = \{B_1, B_2, \ldots, B_c\}$ of sets is called a Δ-system of cardinality c if there exists a set $D \subsetneqq B_i$ $i = 1, \ldots, c$ such that the sets $B_1 - D, \ldots, B_c - D$ are pairwise disjoint. D is called the kernel of the Δ-system.

Theorem 1 (Erdős – Rado [7]). *There exists a function* $\varphi_c(k)$ *such that any family of* $\varphi_c(k)$ *distinct* k*-element sets contains a* Δ*-system of cardinality* c.

An old conjecture of Rado and the second author is that there exists an absolute constant c' such that $\varphi_c(k) < (c \cdot c')^k$. The best existing upper bound – of order about c^k. $k!$ – is due to Spencer [15].

Theorem 2 (Erdős – Ko – Rado [8]). *If* $\mathscr{A}$ *is an* $(p, \{l, l+1, \ldots, k-1\}, k)$*-system of maximal cardinality then for* $n \geqslant n_0(k, l)$ *there exists a set* D *of cardinality* l *such that for every*

$A \in \mathscr{A}$, $D \subseteq A$ *holds. In particular for* $l = 1$ $n_0(k, l) = 2k + 1$ *is the best possible value for* $n_0(k, l)$.

(For $l \geqslant 2$ the best existing upper bound on $n_0(k, l)$ is due to **Frankl** [10]).

Theorem 3 (Deza [1]). *An* $(n, \{l\}, k)$*-system of cardinality more than* $k^2 - k + 1$ *is a* Δ*-system.*

The object of this paper is to generalize Theorems 2 and 3 for (n, L, K)-systems. In the proofs heavy use is made of Theorem 1.

The next four theorems express properties of (n, L, k)-systems.

Troughout we assume $n > n_0(k, \epsilon)$ $\epsilon > 0$.

Let us set $c(k, L) = \max(k - l_1 + 1, l_2^2 - l_2 + 1) + \epsilon$. $\mathscr{A}$ is an (n, L, k)-system.

Theorem 4. *If* $|\mathscr{A}| \geqslant c(k, L) \prod_{i=2}^{r} \frac{n - l_i}{k - l_i}$ *then there exists a set* D *of cardinality* l_1 *such that* $D \subseteq A$ *for every* $A \in \mathscr{A}$.

Theorem 5. *If* $|\mathscr{A}| \geqslant k^2 2^{r-1} n^{r-1}$ *then* $(l_2 - l_1) | (l_3 - l_2) | \ldots$ $\ldots | (l_r - l_{r-1}) | (k - l_r)$.

Theorem 6. $|\mathscr{A}| \leqslant \prod_{i=1}^{r} \frac{n - l_i}{k - l_i}$.

The following result is a generalization of Theorems 4, 5, 6 for (n, L, K)-systems.

Let $K = \{k_1, \ldots, k_s\}$ $k_1 < \ldots < k_s$.

Let us define $K_0 = K \cap \{0, \ldots, l_1\}$, $K_i = \{l_i + 1, \ldots, l_{i+1}\} \cap K$ for $i = 1, \ldots, r - 1$ and $K_r = K \cap \{l_r + 1, \ldots, k_s\}$.

Let us set $k_i^* = \min\{k \mid k \in K_i\}$ for $i = 0, \ldots, r$.

Theorem 7. *Let* $\mathscr{A}$ *be an* (n, L, K)*-system.*

(i) *If* $|\mathscr{A}| > k_s c(k_s, L) \prod_{i=2}^{r} \frac{n - l_i}{k^* - l_i}$ *then there exists a set* D *of cardinality* l_1 *such that* $D \subseteq A$ *for every* $A \in \mathscr{A}$.

(ii) *If* $|\mathscr{A}| > k_s^3 2^{r-1} n^{r-1}$ *then there exists a* $k \in K_r$ *such that* $(l_2 - l_1) | (l_3 - l_2) | \ldots (l_r - l_{r-1}) | (k - l_r)$.

(iii) $|\mathscr{A}| \leqslant \sum_{i=0}^{r} \epsilon_i \prod_{j=1}^{r} \frac{n - l_j}{k_i^* - l_j}$ *where* $\epsilon_i = 0$ *if* $K_i = \phi$, $\epsilon_i = 1$ *otherwise.*

The next theorem is a common generalization of Theorems 4, 6 and a theorem of Hajnal, Rothschild [11].

Theorem 8. *Let* $\mathscr{A}$ *be a family of* k*-element subsets of the* n*-element set* X *such that whenever* $A_1, \ldots, A_{q+1}$ *are* $q + 1$ *different sets belonging to* $\mathscr{A}$ *we can find two of them* A_i, A_j *such that* $|A_i \cap A_j| \in L$ $(q \geqslant 1$ *is fixed). Then*

(i) *there exists a constant* $c = c(k, q)$ *such that*

$$|\mathscr{A}| > (q - 1) \prod_{i=1}^{r} \frac{n - l_i}{k - l_i} + cn^{r-1}$$

implies the existence of sets $D_1, D_2, \ldots, D_s$ *such that for every* $A \in \mathscr{A}$ *there exists an* i $1 \leqslant i \leqslant s$ *satisfying* $D_i \subset A$, $|D_1| = \ldots = |D_s| = l_1$. *Further if* q_i *denotes the maximum number of sets* $A_1, \ldots, A_{q_i}$ *such that for* $1 \leqslant j \leqslant q_i$ $D_i \subset A_j$ *but for* $i' \neq i$ $D_{i'} \not\subset A_j$ *and* $|A_{j_1} \cap A_{j_2}| \notin L$ *for* $1 \leqslant j_1 < j_2 \leqslant q_i$, *then* $\sum_{i=1}^{s} q_i = q$,

(ii) $|\mathscr{A}| \leqslant q \prod_{i=1}^{r} \frac{n - l_i}{k - l_i} + O(n^{r-1})$ $(n > n_0(k, q))$.

In the next theorem we generalize Theorems 4, 5, 6 for the case of t-wise intersections.

Theorem 9. *Let* $\mathscr{A}$ *be a family of* k*-subsets of* X. *Suppose that for any* t *different members* $A_1, \ldots, A_t$ *of* $\mathscr{A}$ $|A_1 \cap \ldots \cap A_t| \in L$. *Then*

(i) *there exists a constant* $c = c(k, t)$ *such that* $|\mathscr{A}| > cn^{r-1}$ *implies the existence of an* l_1*-element set* D *such that* $D \subset A$ *for every* $A \in \mathscr{A}$,

(ii) $|\mathscr{A}| > cn^{r-1}$ *implies* $(l_2 - l_1) | \ldots | (l_r - l_{r-1}) | (k - l_r)$,

(iii) $|\mathscr{A}| \leq (t-1) \prod_{i=1}^{r} \frac{n - l_i}{k - l_i}$ $(n > n_0(k, t))$.

First versions of Theorems 4, 5, 6, 7 was announced in [2], the case $|L| = 2$ was considered in [4].

The proofs of the theorems will appear in the Proceedings of the London Math. Soc.

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