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BIASED POSITIONAL GAMES

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Two players play a game on the complete graph with n vertices. Each move of the first player consists of claiming k previously unclaimed edges, each move of the second player consists of claiming one previously unclaimed edge. The second player's goal is to claim all the edges of some tree on the n vertices, the first player's goal is to prevent the second from doing that. If k is sufficiently large (resp. small) with respect to n then the first (resp. second) player has a win. We prove that the breaking point comes around $k = n/\log n$. In addition, we consider several other games of this kind.

1. Introduction

Two players, who we shall call the Maker and the Breaker, play a game on a multigraph $G = (V, E)$. They take turns, with the Breaker going first, and each of them in his turn claims some previously unclaimed edge of G. The Maker's aim is to claim all the edges of some spanning tree of G; the Breaker's aim is simply to prevent the Maker from achieving his goal. A more general version of this game has been studied by Lehman [7]. His game, generalizing Shannon's "switching game", is played on the elements of a matroid M; the Maker's aim is to claim all the elements of some basis of M. Lehman characterized those matroids on which the Maker can win against every strategy of the Breaker and he described the Maker's winning strategy. His results were complemented by Edmonds [3] who characterized those matroids on which the Breaker can win against every strategy of the Maker and described the Breaker's winning strategy. In the above special case of Lehman's game, the existence of two disjoint spanning trees of G implies the existence of Maker's winning strategy. (An easy proof goes by induction on the number of vertices of G.) On the other hand, Tutte [9] and Nash-Williams [8] proved independently of each other that the nonexistence of two disjoint spanning trees of G is equivalent to the existence of a set A of edges whose deletion splits G into at least $\frac{1}{2}(|A|+3)$ components. When such a set exists, the Breaker wins simply by claiming edges form A as long as there remain any. (The theorem of Tutte and Nash-Williams is more general. It asserts that G has k pairwise disjoint spanning trees if and only if there is no set A of edges whose deletion splits G into more than $1+|A|/k$ components. This theorem has then been generalized in the context of matroids by Edmonds [2]. Edmonds' theorem is accompanied by an efficient algorithm which, in the special case of multigraphs, terminates by constructing either the set A or the k pairwise disjoint spanning trees.)

When played on a large complete graph with n vertices, Lehman's game is overwhelmingly in favor of the Maker. We shall make up for this handicap by allowing the Breaker to claim many edges per move. More precisely, we shall choose a positive integer b and let each move of the Breaker consist of claiming b previously unclaimed edges. Clearly, if b is large with respect to n then the first player has a win; if b is small with respect to n then the second player has a win. The following heuristic argument suggests the breaking point ought to come around $n/\log n$. The duration of the game allows for approximately $n^2/2b$ Maker's edges. In particular, if $b = n/2c \log n$, then the Maker will have the time to create a graph with $cn \log n$ edges. A *random* graph with n vertices and $cn \log n$ edges is almost certainly connected for $c > 1/2$ and almost certainly disconnected for $c < 1/2$. (That is the statement of an unpublished theorem by Erdös and Whitney, which has been sharpened by Erdös and Rényi [4].) We shall prove that the breaking point does indeed come around $b = n/\log n$; more precisely, it is between $n/(4+\varepsilon)\log n$ and $(1+\varepsilon)n/\log n$ for all sufficiently large n.

Lehman's switching game belongs to the general class of "positional games" studied by Hales and Jewett [6], Erdös and Selfridge [5], Berge [1] and others. Of course, every positional game can be made biased by allowing one (or both) of the players to claim more than one position per move. Those games that we shall study here fall into the following general pattern. They are played on some hypergraph H. The two players, the Breaker and the Maker, take turns in claiming previously unclaimed vertices; the Breaker has the first move. On each move, the Breaker claims exactly b vertices and the Maker claims exactly m vertices. The Maker's aim is to claim all the vertices of some edge; the Breaker's aim is simply to prevent the Maker from doing so. In the next section, we shall solve an extremely simple game of this kind: $b = 1$, the edges of H are pairwise disjoint and almost equal in size. (The solution to this "box game" will be handy in two different contexts when we shall discuss the biased version of Lehman's game.) In the following three sections, we shall study games for which $m = 1$ and the vertices of H are the edges of a complete graph. For the edges of H, we shall successively take spanning trees, hamiltonian cycles and complete subgraphs of prescribed size. The last of these games generalizes quite naturally into the context of r-graphs.

2. The box game

We shall say that a hypergraph H is of *type* (k, t) if its edges $A_1, A_2, \ldots, A_k$ are pairwise disjoint and such that $\sum |A_i| = t$. If, in addition, the edges are "almost equal" in size (that is, if $|A_i|$ and $|A_j|$ differ by at most one for all choices of i and j) then we shall say that H is *canonical* of type (k, t). The "box game" $B(k, t, m)$ is played on a canonical hypergraph of type (k, t). The two players take turns (with the Breaker having the first move) in claiming previously unclaimed vertices of H. On each move, the Breaker claims only one vertex whereas the Maker claims m vertices. The Maker's aim is to claim all the vertices of some edge A_i.

It is convenient to think of the actions of the two players in the following way. By claiming a vertex from some edge A_j, the Breaker removes this edge from H; by claiming a vertex v from some edge A_i, the Maker replaces this edge by $A_i - \{v\}$. Hence after each move of the Maker, and the counter-move of the Breaker, the hypergraph H reduces into a new hypergraph H^*. Note that the Maker can always play in such a way that the new hypergraph H^* is again canonical.

In order to present a solution of $B(k, t, m)$, we shall define $f(1, m) = 0$ and

$$f(k, m) = \lfloor k(f(k-1, m) + m)/(k-1) \rfloor$$

for all positive integers k and m such that $k \geq 2$. It is not difficult to verify that

$$(m-1)k \sum_{i=1}^{k-1} \frac{1}{i} \leq f(k, m) \leq mk \sum_{i=1}^{k-1} \frac{1}{i}.$$

Theorem 2.1. *The Maker has a winning strategy on $B(k, t, m)$ if and only if $t \leq f(k, m)$.*

Proof. To prove the "if" part, we shall use induction on k. Responding to Breaker's move, the Maker can create a canonical hypergraph H^* of type $(k-1, t^*)$ such that $t^* \leq t - \lfloor t/k \rfloor - m$. Since the right-hand side of this inequality is at most $f(k-1, m)$, we are done.

To prove the "only if" part, we shall again use induction on k. This time, however, we shall prove a slightly stronger statement: if $t > f(k, m)$, then the Breaker has a winning strategy for the box game played on an *arbitrary*, not necessarily canonical, hypergraph of type (k, t). The strategy consists of removing, at each move, the smallest available edge. No matter how the Maker responds, the resulting hypergraph H^* will be of type $(k-1, t^*)$ such that $t^* \geq t - \lfloor t/k \rfloor - m$. Since the right-hand side of this inequality is strictly greater than $f(k-1, m)$, we are done again.

3. Spanning trees

By $T(n, b)$, we shall denote the biased version of Lehman's game described in the introduction: the Breaker claims b edges per move and the Maker wants a spanning tree.

Theorem 3.1. *If ε is positive, if $n > n(\varepsilon)$ and if $b > (1+\varepsilon)n/\log n$, then the Breaker has a winning strategy for $T(n, b)$.*

Proof. The Breaker proceeds in two stages. In the first stage, he will claim all the edges of some clique C with k vertices such that $k = \lceil n/2 \log n \rceil$ and such that none of the Maker's edges has an endpoint in C. In the second stage, he will claim all the remaining edges incident with some $v \in C$, thereby preventing the Maker from joining v to other vertices. The first stage lasts no more than k moves. During the first $t-1$ moves $(1 \leq t \leq k)$, the Breaker has created a clique C with $t-1$ vertices such that none of the Maker's edges has an endpoint in C. At the moment, there are exactly $t-1$ Maker's edges; hence there are vertices $u, v \notin C$ which are incident with none of the Maker's edges. On his t^{th} move, the Breaker claims the edge uv and all the edges joining u and v to vertices from C, thereby enlarging C by two vertices. The Maker may, in his turn, eliminate one vertex from C by claiming an edge incident with that vertex. Nevertheless, the size of C will still have increased by at least one vertex. In the second stage, the Breaker thinks of every set of edges joining some $u \in C$ to all $v \notin C$ as the edge of a canonical hypergraph of type (k, t) with $t = k(n-k)$. He then plays the box game on this hypergraph, pretending to be the Maker. By Theorem 2.1, the inequality

$$t \leq (b-1)k \sum_{i=1}^{k-1} \frac{1}{i}$$

is sufficient to guarantee his victory. And, as can be seen after the appropriate substitutions, this inequality is indeed satisfied.

Our next theorem provides a winning strategy for the Maker. We shall describe it in the more general context of the game $T(G, b)$ which is played on the edges of a multigraph G.

Theorem 3.2. *Let $G = (V, E)$ be a multigraph with n vertices and let b be a positive integer. Assume that for every subset S of V and for every vertex v such that $v \notin S$ and $|S| \geq n/2$, more than*

$$2b \sum_{i=1}^{n} \frac{1}{i}$$

edges of G join v to vertices of S. Then the Maker has a winning strategy for $T(G, b)$.

Proof. For each $u \in V$, we shall denote by $d_k(u)$ the number of edges of G that are incident with u and have been claimed by the Breaker during his first k moves. For each subset S of V, we shall define

$$d_k(S) = \min_{u \in S} d_k(u).$$

We shall denote by M_k the subgraph of G consisting of the edges claimed by the Maker within his first k moves. The strategy is quite simple: after the Breaker's k^{th} move, the Maker finds a component A of M_{k-1} which maximizes $d_k(C)$ over all components C of M_{k-1}. We guarantee that there will be an unclaimed edge joining A to $V - A$. (This will be proved below). The Maker claims this edge and the Breaker resumes the play. Making his $(n-1)^{\text{st}}$ move in this fashion, the Maker will complete a tree on n vertices and win.

In order to prove that this strategy can be carried out, we shall denote the following hypothesis by $H(k)$: for every integer t such that $1 \leq t \leq n-k+1$ and for every choice of components $C_1, C_2, \ldots, C_t$ of M_{k-1}, we have

$$\sum_{i=1}^{t} d_{k-1}(C_i) \leq 2bt \sum_{i=t+1}^{n} \frac{1}{i}.$$

Note that $H(1)$ holds trivially since $d_0(u)=0$ for every vertex u. Assuming $H(k)$, we shall derive the existence of the unclaimed edge joining A to $V-A$ and establish the validity of $H(k+1)$.

To begin with, we have

$$\sum_{i=1}^{t} d_k(C_i) \leq 2b + \sum_{i=1}^{t} d_{k-1}(C_i) \leq 2bt \sum_{i=t}^{n} \frac{1}{i}, \tag{1}$$

for every choice of components $C_1, C_2, \ldots, C_t$ of M_{k-1}. In particular, we have

$$d_k(C) \leq 2b \sum_{i=1}^{n} \frac{1}{i},$$

for every component C of M_{k-1}. Hence every component of M_{k-1} includes at least one vertex u such that

$$d_k(u) \leq 2b \sum_{i=1}^{n} \frac{1}{i}.$$

From this fact, and from the hypothesis of our theorem, it follows that

> for every component C of M_{k-1} and for every subset
> S of V such that $|S| \geq n/2$ and $C \cap S = \emptyset$, there (2)
> is an unclaimed edge uv such that $u \in C$, $v \in S$.

Now, the existence of an unclaimed edge joining A to $V-A$ follows immediately: if $|A| \leq n/2$, then we use (2) with $C = A$ and $S = V - A$, if $|A| > n/2$, then we use (2) with $S = A$.

It remains to verify $H(k+1)$. For each component C of H_{k-1}, we shall choose a vertex u such that $d_k(u) = d_k(C)$; we shall call this vertex the *root* of C. By (1), we have

$$\sum_{i=1}^{t} d_k(u_i) \leq 2bt \sum_{i=t}^{n} \frac{1}{i} \tag{3}$$

for every choice of roots $u_1, u_2, \ldots, u_t$, such that $1 \leq t \leq n-k+1$. Let w denote the root of A; by maximality of $d_k(A)$, we have

$$d_k(w) \geq d_k(u), \quad \text{for every root } u. \tag{4}$$

Now, let us assume $H(k+1)$ false. Then we have

$$\sum_{i=1}^{t-1} d_k(C_i) > 2b(t-1) \sum_{i=t}^{n} \frac{1}{i}, \tag{5}$$

for some integer t such that $1 \leq t-1 \leq n-k$ and for some choice of components

$C_1, C_2, \ldots, C_{t-1}$ of M_k. Note that each C_i includes exactly one root u_i such that $u_i \neq w$; by (5), we have

$$\sum_{i=1}^{t-1} d_k(u_i) \geq \sum_{i=1}^{t-1} d_k(C_i) > 2b(t-1)\sum_{i=t}^{n} \frac{1}{i}.$$

Setting $u_t = w$ and using (4) we obtain

$$\sum_{i=1}^{t} d_k(u_i) \geq \frac{t}{t-1}\sum_{i=1}^{t-1} d_k(u_i) > 2bt\sum_{i=t}^{n} \frac{1}{i}$$

contradicting (3). Hence $H(k+1)$ is valid and the proof is completed.

Corollary 3.3. *If ε is positive, if $n > n(\varepsilon)$ and if $b < n/(4+\varepsilon)\log n$ then the Maker has a winning strategy for $T(n, b)$.*

Remark. Bondy observed that there are multigraphs G with an arbitrarily large number of pairwise disjoint spanning trees and yet such that the Breaker can win even $T(G, 2)$. The simplest example is the path of length n, each of whose edges has multiplicity s. For this G, the Breaker can view $T(G, 2)$ as the box game $B(n-1, s(n-1), 2)$, with himself in the role of the Maker. If

$$s \leq \sum_{i=1}^{n-2} \frac{1}{i},$$

then, by Theorem 2.1, he has a win. At the same time, however, G has s pairwise disjoint spanning trees.

4. Hamiltonian cycles

By $H(n, b)$, we shall denote the game which differs from $T(n, b)$ in only one respect: the Maker's aim is to claim all edges of some hamiltonian cycle. What is the largest $f(n)$ such that the Maker has a winning strategy for $H(n, f(n))$? In this section, we shall prove that $f(n) \geq 1$ for all sufficiently large n. It is not unlikely that $f(n) \to \infty$, as $n \to \infty$; however, we cannot even prove that $f(n) \geq 2$ for all sufficiently large n.

Theorem 4.1. *For all sufficiently large n, the Maker has a winning strategy for $H(n, 1)$.*

Proof. The strategy proceeds in three stages. The first stage is the simplest. It lasts exactly $m = \lfloor n/3 \rfloor$ moves; by the end of that stage, the Maker will have claimed all the edges of some cycle C_0 whose length is between $n/4$ and $n/3$. In his first $k-1$ moves ($1 \leq k < m$), the Maker has claimed all the edges of some path $u_1 u_2 \cdots u_k$. In his k^{th} move, he claims a previously unclaimed edge $u_k v$ such that $v \neq u_i$ for all i;

on his m^{th} move, he claims a previously unclaimed edge u_iu_j such that $1 \leq i < n/24$ and $7n/24 < j \leq m$.

By the end of the first stage, at most eight vertices outside C_0 have at least $n/24$ Breaker-neighbours (that is, neighbours in the Breaker's graph) in C_0. Such vertices will be called *dangerous*. For each dangerous vertex v, the Maker will use four moves to enlarge his current cycle C into a cycle C^* containing v. Hence the entire second stage will last at most 32 moves. In order to construct C^*, the Maker first finds three vertices w_1, w_2, w_3 outside C such that the edges vw_1, vw_2, vw_3 are unclaimed and such that each w_i has at most $n/24 - 3$ Breaker-neighbours in C. (This can be done for the following reason. The vertex v has Breaker-degree at most $n/3 + 32$; the cycle C has length at most $n/3 + 32$. Hence there are at least $n/3 - 65$ vertices w outside C such that vw is unclaimed. If all but possibly two of them had at least $n/24 - 2$ Breaker-neighbours in C, then the Breaker's graph would have at least $(n/3 - 65)(n/24 - 2) > n/3 + 32$ edges, a contradiction.) The Maker claims vw_i and, after the Breaker's next move, he claims another vw_j. After the following move of the Breaker, each of the two vertices w_i and w_j still has less than $n/24$ Breaker-neighbours in C. An easy averaging argument shows that there must be three consecutive vertices u_1, u_2, u_3 on C such that none of the edges w_iu_k and w_ju_k has been claimed. The Maker claims w_iu_2 and, on his next move, either w_ju_1 or w_ju_3.

When the second stage is over, every vertex outside the Maker's current cycle C has at most $n/24 + 32 < n/18$ Breaker-neighbours in C. If, at some time during the three stages, a vertex v outside C will have at least $n/18$ Breaker-neighbours in C then we shall call v *outstanding*. The Maker proceeds in steps, each step consisting of two moves. He selects a vertex v outside C (preferably an outstanding one if there is any) and in two moves, he enlarges C into a cycle C^* containing v. Hence the entire game will be over in fewer than $2n$ moves. Consequently, at most 72 outstanding vertices will make their appearance; each of them will wait no more than 144 moves for the inclusion into C. During that waiting time, the number of its Breaker-neighbours in C may grow to at most $n/18 + 144 < n/12$. We conclude that at every point of the third stage, every vertex v outside C will have fewer than $n/12$ Breaker-neighbours in C. An easy averaging argument shows that there will be three consecutive vertices u_1, u_2, u_3 on C such that all three edges vu_k are unclaimed. In order to enlarge C into C^*, the Maker claims first the edge u_2v and, on his next move, either u_1v or u_3v.

5. Complete subgraphs

By $C(n, 2, 3, b)$ we shall denote the game which differs from $T(n, b)$ and $H(n, b)$ in the Maker's aim: he wants to claim all three edges of some triangle.

Theorem 5.1. *If $b < (2n + 2)^{1/2} - 5/2$ then the Maker has a winning strategy for*

$C(n, 2, 3, b)$. On the other hand, if $b \geq 2n^{1/2}$ then the Breaker has a winning strategy for $C(n, 2, 3, b)$.

Proof. The Maker chooses a vertex u and claims as many edges incident with u as possible. When all the edges incident with u have been claimed, he finds an unclaimed edge vw such that both uv and uw have been claimed by himself; claiming vw, he wins. Of course, we have to show that he can always carry out this plan. Let us assume that he cannot. Then, at some point of the game, he has claimed exactly d edges uv; all the remaining $n-1-d$ edges uv, as well as $d(d-1)/2$ edges vw, have been claimed by the Breaker. Hence

$$(n-1-d)+d(d-1)/2 \leq (d+1)b.$$

However, this inequality has no solution d as long as

$$4b^2+20b-8n+17<0$$

which is the case when $b<(2n+2)^{1/2}-5/2$.

Next, let us describe the Breaker's strategy. If the Maker has just claimed some edge uv then the Breaker will claim $\lfloor n^{1/2} \rfloor$ edges incident with u and $\lfloor n^{1/2} \rfloor$ edges incident with v. Hence the degree of w in the Maker's graph will never exceed $\lfloor n^{1/2} \rfloor + 1$. Of course, the Breaker must prevent the Maker from completing a triangle. Hence he must claim immediately every edge xy such that, for some vertex z, both of the edges xz and yz have been claimed by the Maker. Since he does so systematically, the only dangerous edges xy at the moment have either the form uy such that vy is the Maker's edge or the form vy such that uy is the Maker's edge. Since the Maker-degree of u and v does not exceed $\lfloor n^{1/2} \rfloor + 1$, there are at most $\lfloor n^{1/2} \rfloor$ dangerous edges of each kind. Hence the Breaker will be able to claim them all.

The concept of $C(n, 2, 3, b)$ generalizes into that of $C(n, r, k, b)$. This game is played on the edges of a complete r-graph with n vertices (each edge being an r-subset of the fixed n-set of vertices). The Breaker claims b edges per move; the Maker wants to claim all the $\binom{k}{r}$ edges of some complete r-graph with k vertices.

Theorem 5.2. *For every choice of positive integers r, k and b such that $k \geq r$ there is a positive integer $n(r, k, b)$ with the following property: for every $n \geq n(r, k, b)$, the Maker has a winning strategy for $C(n, r, k, b)$.*

Proof. Trivially, $n(1, k, b) = k(b+1)$ and $n(r, r, b)$ is the smallest integer x such that $\binom{x}{r} > b$. By double induction on r and k, it will suffice to prove that the Maker has a winning strategy for $C(n, r, k, b)$ as long as

$$n \geq 1 + n(r-1, n(r, k-1, b), b). \tag{6}$$

The Maker proceeds in two stages. In the first stage, selecting some vertex v, he will claim only those edges S for which $v \in S$. He will interpret each of them as the

$(r-1)$-set $S-\{v\}$; similarly, he will interpret each of the Breaker's edges T as some $(r-1)$-set T^* such that $v \notin T^*$, $T^* \subset T$. Altogether, he will interpret the first stage as the game $C(n-1, r-1, n(r, k-1, b), b)$. By (6), he has a winning strategy for this game; he will indeed follow that strategy. By the end of the first stage, there will be a set X of vertices such that $|X| = n(r, k-1, b)$, $v \notin X$ and such that, for each $(r-1)$-subset A of X, the edge $A \cup \{v\}$ has been claimed by the Maker. In the second stage, the Maker will simply play $C(|X|, r, k-1, b)$, restricting his choice of edges to subsets of X.

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