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A Note On Ingham's Summation Method

P. ERDÖS

The Hungarian Academy of Sciences, Budapest, Hungary

AND

S. L. SEGAL

Department of Mathematics, University of Rochester, Rochester, New York 14627

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Ingham's (nonregular) summation method (I) is closely connected with prime number theory. An easy limitation theorem for (I) (observed by Hardy) is if Σc_n is summable (I) then $c_n = o(\log\log n)$. We show this result to be best possible.

Ingham's summation method (I) [2] (also discovered independently by Wintner [6]) may be defined as follows: A series Σc_n will be said to be summable (I) to A if

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \sum_{d|n} dc_d = \lim_{x\to\infty} \frac{1}{x} \sum_{d\leq x} dc_d \left[\frac{x}{d}\right] = \lim_{x\to\infty} \frac{1}{x} \sum_{d\leq x} \sum_{m\leq x/d} mc_m = A,$$

where $[x]$ as usual is the greatest integer in x and the three forms of the limit are clearly equal. (I)-summability is closely connected with prime number theory, and was used by Ingham to give an original proof of the Prime Number Theorem (for further details of such connections see [2; 1: Appendix IV; and 5]).

Let

$$I(x) = \frac{1}{x} \sum_{n\leq x} \sum_{d|n} dc_d .$$

Then, if $I(x) = A + o(1)$, multiplication by x, subtraction, and Möbius inversion show that:

$$\text{If } \Sigma c_n \text{ is } (I)\text{-summable, then } c_n = o(\log\log n),$$

as observed by Hardy [1, Theorem 265]. We show this is best possible.

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THEOREM. *There exists a series Σa_n which is (I)-summable and for which $a_n/\log\log n \to 0$ arbitrarily slowly as $n \to \infty$.*

In the proof p will always denote prime numbers, and $\mu(n)$ and $\theta(n)$ their usual meanings in prime number theory.

Proof. Define the sequence $\{n_k\}$ by

$$n_0 = 1; \qquad n_1 = 5; \qquad \text{for } k \geqslant 2, \qquad n_k = \prod_{p \leqslant n_{k-1}} p. \tag{1}$$

Let

$$\mathscr{S} = \{n_k/d : d \text{ divides } n_k,\ 1 \leqslant d \leqslant n_{k-1},\ k \in \mathbb{N}\}. \tag{2}$$

Let $S(n)$ be the characteristic function of $\mathscr{S}$; that is,

$$\begin{aligned} S(n) &= 1, \qquad n \in \mathscr{S}, \\ &= 0, \qquad \text{otherwise.} \end{aligned} \tag{3}$$

Let $\epsilon(r)$ be a positive function tending monotonically to 0 arbitrarily slowly as $r \to \infty$.

Let the sequence a_n be defined by

$$b_r = \frac{1}{r} \sum_{d \mid r} d a_d\,, \tag{4}$$

and define b_r by

$$b_r = \mu(r)\,\epsilon(r)\,S(r) - \mu(r-1)\,\epsilon(r-1)\,S(r-1). \tag{5}$$

We may note that since for $k \geqslant 2$, $n_k = e^{\theta(n_{k-1})}$, and $\theta(n_{k-1}) > 2\theta(n_{k-2})$ for $k \geqslant 3$ (cf. [3]), that

$$n_k/n_{k-1} > n_{k-1} \tag{6}$$

and so each element of $\mathscr{S}$ has a unique representation as n_k/d, $1 \leqslant d \leqslant n_{k-1}$.

Clearly Σb_r converges to 0 and so $\sum_{r \leqslant x} r b_r = o(x)$ as $x \to \infty$, whence Σa_n is (I)-summable by (4). On the other hand from (4) and (5), by Möbius Inversion,

$$\begin{aligned} a_{n_l} &= \sum_{d \mid n_l} \frac{\mu(d)}{d}\, \mu\left(\frac{n_l}{d}\right) \epsilon\left(\frac{n_l}{d}\right) S\left(\frac{n_l}{d}\right) \\ &\quad - \sum_{d \mid n_l} \frac{\mu(d)}{d}\, \mu\left(\frac{n_l}{d} - 1\right) \epsilon\left(\frac{n_l}{d} - 1\right) S\left(\frac{n_l}{d} - 1\right) \\ &= \Sigma_1 - \Sigma_2\,, \qquad \text{say.} \end{aligned} \tag{7}$$

For Σ_1, we have, since n_l is square-free,

$$\Sigma_1 = \mu(n_l) \sum_{d \mid n_l} \frac{\mu^2(d)}{d} \epsilon\left(\frac{n_l}{d}\right) S\left(\frac{n_l}{d}\right).$$

Hence, by definition of S, and ϵ,

$$\mu(n_l)\, \Sigma_1 \geqslant \sum_{\substack{d \mid n_l \\ d \leqslant n_{l-1}}} \frac{\mu^2(d)}{d} \epsilon\left(\frac{n_l}{d}\right) \geqslant \epsilon(n_l) \sum_{\substack{d \mid n_l \\ d \leqslant n_{l-1}}} \frac{\mu^2(d)}{d}. \tag{8}$$

But if $d \leqslant n_{l-1}$ and square-free then d is a distinct product of primes $\leqslant n_{l-1}$ and so $d \mid n_l$. Hence (8) yields

$$\begin{aligned} \mu(n_l)\, \Sigma_1 &\geqslant \epsilon(n_l) \sum_{d \leqslant n_{l-1}} \frac{\mu^2(d)}{d} = \frac{6}{\pi^2} \epsilon(n_l) \log n_{l-1} + O(1) \\ &\sim \frac{6}{\pi^2} \epsilon(n_l) \log \theta(n_{l-1}) \\ &= \epsilon'(n_l) \log\log n_l \end{aligned} \tag{9}$$

as $l \to \infty$, where $\epsilon'(n_l) \to 0$ arbitrarily slowly as $l \to \infty$.

To estimate Σ_2 we need to compute when $(n_l/d) - 1$ can be of the form n_k/r, $1 \leqslant r \leqslant n_{k-1}$. There are three possible cases.

Case I. $k \geqslant l + 1$.

Then, since $\{n_k/n_{k-1}\}$ is clearly a monotone increasing sequence, we have, if this case should hold,

$$n_l > \frac{n_l}{d} - 1 = \frac{n_k}{r} \geqslant \frac{n_k}{n_{k-1}} \geqslant \frac{n_{l+1}}{n_l},$$

contradicting (6).

Case II. $k = l$.

Then, if $(n_l/d) - 1 = n_l/r$, $r \geqslant d + 1$, and

$$n_l = \frac{dr}{r-d} \leqslant \left(\frac{1}{r-1} - \frac{1}{r}\right)^{-1} = r(r-1) < (n_{l-1})^2,$$

again a contradiction to (6).

Case III. $k \leqslant l - 1$.

Then, if $(n_l/d) - 1 = n_k/r \leqslant n_{l-1}$, we have $d \geqslant n_l/(n_{l-1} + 1)$. Since only in Case III are there possibly nonzero values of S in Σ_2, we have

$$|\Sigma_2| \leqslant \sum_{\substack{d \mid n_l \\ d \geqslant n_l/(n_{l-1}+1)}} \frac{1}{d} \epsilon\left(\frac{n_l}{d} - 1\right) \leqslant K \frac{n_{l-1}}{n_l} \sum_{\substack{d \mid n_l \\ d \geqslant n_l/(n_{l-1}+1)}} 1, \tag{10}$$

where K is a positive constant. However,

$$\sum_{\substack{d \mid n_l \\ d \geqslant n_l/(n_{l-1}+1)}} 1 = \sum_{\substack{d \mid n_l \\ d \leqslant n_{l-1}+1}} 1 = O(n_{l-1}).$$

Hence by (10), we get

$$|\Sigma_2| = O\left(\frac{(n_{l-1})^2}{n_l}\right) = O(1), \quad \text{as} \quad l \to \infty. \tag{11}$$

Putting (11) and (9) into (7) we get

$$|a_{n_l}| = |\mu(n_l)\, a_{n_l}| \geqslant (1 + o(1))\, \epsilon'(n_l) \log\log n_l$$

as $l \to \infty$, which proves the theorem.

On the other hand, as noted earlier, we must have $a_{n_l} = o(\log\log n_l)$.

Omitting the function ϵ we have an example of an (I)-bounded series (not (I)-summable) for which $a_n = \Omega(\log\log n)$.

It is perhaps worth making two further remarks.

(1) Although it was known to Ingham and Wintner that (I)-summability did not imply convergence (see [2, p. 180]; [6 p. 13]); the above is apparently the first explicit example of an (I)-summable series which is not convergent. The effective construction of such an example is a question apparently raised in Ingham's posthumous papers.

(2) With $\{a_n\}$ as constructed in the theorem, and $\sum_{n \leqslant x} \sum_{d \mid n} d a_d = xI(x)$, we have an example of a function $I(x)$ such that $I(x) \to 0$ as $x \to \infty$, but $\sum_{d \leqslant x} (\mu(d)/d)\, I(x/d) \not\to 0$ as $x \to \infty$ (compare [4]).

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