A CLASS OF RAMSEY-FINITE GRAPHS

S. A. Burr
A.T.&L. Long Lines

P. Erdös
Hungarian Academy of Sciences

R. J. Faudree and R. H. Schelp
Memphis State University

Introduction

Let F, G and H be finite, undirected graphs without loops or multiple edges. Write $F \longrightarrow (G,H)$ to mean that if the edges of F are colored with two colors, say red and blue, then either the red subgraph of F contains a copy of G or the blue subgraph contains a copy of H. The class of all graphs F (up to isomorphism) such that $F \longrightarrow (G,H)$ will be denoted by $\mathcal{R}'(G,H)$. This class has been studied extensively, for example the generalized Ramsey number $r(G,H)$ is the minimum number of vertices of a graph in $\mathcal{R}'(G,H)$.

PROC. 9TH S-E CONF. COMBINATORICS, GRAPH
THEORY, AND COMPUTING, pp. 171-180.

A graph F is (G,H)-minimal if $F \in R'(G,H)$ but no proper subgraph of F is in $R'(G,H)$. Denote the class of (G,H)-minimal graphs by $R(G,H)$. Since each graph in $R'(G,H)$ has a subgraph which is in $R(G,H)$, it is natural to study the smaller class $R(G,H)$. A pair (G,H) is <u>Ramsey-finite</u> if $R(G,H)$ is finite. Otherwise the pair (G,H) is <u>Ramsey-infinite</u>. There are large classes of graphs which are Ramsey-infinite. For example, Nesetril and Rödl, [6], have shown that the pair (G,H) is Ramsey-infinite if both G and H are 3-connected or if G and H are forests neither of which is a union of stars. By comparison the class of pairs of graphs (G,H) which are known to be Ramsey-finite is small. Trivially the pair (K_2,H) is Ramsey-finite since $R(K_2,H) = \{H\}$. If G and H are both disjoint unions of edges then (G,H) is Ramsey-finite, [4], or if G and H are stars with an odd number of edges then (G,H) is Ramsey-finite [3].

In this paper a proof that the pair (mK_2,H) is Ramsey-finite for $m \geq 1$ and H an arbitrary graph will be given. This will be accomplished by giving an upper bound on the number of edges of any graph in $R(mK_2,H)$.

The Main Result

The notation used within this paper will be fairly standard. The edge set and vertex set of a graph G will be denoted by E(G) and V(G) respectively. As usual

K_n, C_n and $K_{1,n}$ will denote a complete graph on n vertices, a cycle with n vertices, and a star with n edges respectively. The graph consisting of n disjoint copies of a graph G will be written nG. If H is a subgraph of G, then $G-H$ will be the graph obtained from G by deleting the edges of H. If v is a vertex of G, then the graph obtained by deleting v and its incident edges will be denoted by $G-v$. Notation not specifically mentioned will follow Harary [5].

The central result of the paper is the following theorem and its corollary.

<u>Theorem</u>: Let G be an arbitrary graph on n vertices and m a positive integer. Then for $F \in \mathcal{R}(mK_2,G)$,

$$|E(F)| \leq \sum_{i=1}^{b} n^i$$

where $b = (m-1)(\binom{2m-1}{2} + 1) + 1$.

An immediate consequence of this is the following.

<u>Corollary</u>: For m a positive integer and G an arbitrary graph, the pair (mK_2,G) is Ramsey-finite.

The next two lemmas will be needed in the proof of the theorem.

<u>Lemma 1</u>: If $mK_2 \not\subseteq H$, then

$$|E(H)| \leq \max\{\binom{2m-1}{2}, (m-1)\Delta(H)\}$$

<u>Proof</u>: With no loss of generality it can be assumed that $(m-1)K_2 \subseteq H$. This implies that H is isomorphic to a subgraph of

$$K_s + (\sum_{i=1}^{\ell} K_{2n_i+1})$$

where $s \geq \ell$ and $s + \sum_{i=1}^{\ell} n_i = m - 1$, ([1]).

Therefore $|E(H)| \leq s\Delta(H) + \sum_{i=1}^{\ell} \binom{2n_i+1}{2}$.

Since $\binom{a}{2} + \binom{b}{2} \leq \binom{a+b-1}{2}$ for positive integers a and b, $\sum_{i=1}^{\ell} \binom{2n_i+1}{2} \leq \binom{2(m-s-1)+1}{2}$. If $f(s) = s\Delta(H) + \binom{2(m-s-1)}{2}$, then $|E(H)| \leq f(s)$. Since $f(0) = \binom{2m-1}{2}$ and $f(m-1) = (m-1)\Delta(H)$, $|E(H)| \leq \max\{\binom{2m-1}{2}, (m-1)\Delta(H)\}$

Lemma 2: If $mK_2 \nsubseteq H$, then each vertex of H of degree at least $2m - 1$ is contained in any maximal matching of H.

Proof: Let M be a maximal matching of H. By assumption M has at most $2m - 2$ vertices. If v is a vertex of H of degree at least $2m - 1$, then there is a vertex w of H not in M which is adjacent to v. Thus v is a vertex in M, for otherwise $\{vw\} \cup M$ is a matching of H which properly contains M.

Proof of Theorem: Suppose to the contrary that

$$F \in R(mK_2, G) \quad \text{and} \quad |E(F)| > \sum_{i=1}^{b} n^i.$$

It will be shown that this leads to a contradiction.

Let e be an edge of F. Since $F \in R(mK_2, G)$,

$$F - e \not\rightarrow (mK_2, G).$$

Therefore the edges of $F - e$ can be colored with the colors red and blue such that there is no red mK_2 or blue G. Among all such colorings, select a red subgraph with a maximal number of edges. Denote this graph by S_e. The maximiality of S_e and Lemma 2 imply that if v is a vertex of degree at least $2m - 1$ in S_e, then every edge of F incident to v is also in S_e. Since $F \longrightarrow (mK_2, G)$, there is a copy of G in F, say G_e, such that $E(G_e) \cap E(S_e) = \phi$. Clearly $e \in E(G_e)$. In fact note that for any copy G' of G in F,

$$E(G') \cap (\{e\} \cup E(S_e)) \neq \phi.$$

Let $t = |E(F)| \geq \sum_{i=0}^{b} n_i$. Denote the edges of F by $\{e_1, e_2, \ldots, e_t\}$. For each $i \in T_0 = \{1, 2, \ldots, t\}$ there is a triple (e_i, G_i, S_i) where $S_i = S_{e_i}$ and $G_i = G_{e_i}$. It will next be shown that there is a subset $T_{b-1} \subseteq T_0$ with $|T_{b-1}| \geq n + 1$ such that $S_i = S_j$ for all $i, j \in T_{b-1}$ and each S_i is a union of $m - 1$ stars.

Consider the graph G_1. For each $i \in T_0$, $E(G_1) \cap (\{e_i\} \cup E(S_i)) \neq \phi$. Thus $E(G_1) \cap E(S_i) \neq \phi$ if $e_i \notin E(G_1)$. Since G_1 has n edges, there are at least $t - n$ different i's such that $E(G_1) \cap E(S_i) \neq \phi$. In fact there is an edge f_1 of G_1 which is contained in at least $(t-n)/n$ of the graphs S_i. Let

$$T_1 = \{i \in T_0 : f_1 \in E(S_i)\}.$$

Then $|T_1| \geq \sum_{i=0}^{b-1} n^i$.

For some fixed $k \in T_1$, consider the edge e_k and corresponding graph G_k. The edge f_1 is not in $E(G_k)$ since $E(G_k) \cap E(S_k) = \phi$. Using the graph G_k just as G_1 was used in the previous argument, one obtains an edge f_2 with the same properties as f_1. Hence if

$$T_2 = \{i \varepsilon T_1 : f_2 \varepsilon E(S_i)\},$$

then

$$|T_2| \geq \{(|T_1|-n)/n\} \geq \sum_{i=0}^{b-2} n^i.$$

A repetition of this argument yields a set of distinct edges $\{f_1, f_2, \ldots, f_{b-1}\}$ and sets $T_j = \{i \varepsilon T_{j-1} : f_j \varepsilon E(S_i)\}$ such that $|T_j| \geq (|T_{j-1}|-n)/n$ for $1 \leq j \leq b-1$.

Thus $|T_j| \geq \sum_{i=0}^{b-j} n^i$, and in particular

$$|T_{b-1}| \geq \sum_{i=0}^{1} n^i = n + 1.$$

Let $r = \binom{2m-1}{2} + 1$. For each k, $0 \leq k \leq m - 2$, let L_k be the graph spanned by the edges

$$\{f_{kr+1}, f_{kr+2}, \ldots, f_{(k+1)r}\}.$$

Therefore L_k has $\binom{2m-1}{2} + 1$ edges. Also $L_k \subseteq S_i$ for any $i \varepsilon T_{b-1}$, so L_k does not contain a subgraph isomorphic to mK_2. Thus Lemma 1 implies that L_k has a vertex, say w_k, of degree at least $2m - 1$. Therefore for any $i \varepsilon T_{b-1}$, S_i conatins the $m - 1$ vertices $w_0, w_1, \ldots, w_{m-2}$, each of which has degree at least $2m - 1$ in S_i. Let S be the subgraph of F spanned by the edges of F incident to at least one of the vertices of $\{w_0, w_1, \ldots, w_{m-2}\}$. Since each S_i was chosen with a maximal number of edges, Lemma 2 implies that $S_i = S$ for all $i \varepsilon T_{b-1}$.

For some fixed $k \varepsilon T_{b-1}$, consider the triple (e_k, G_k, S_k). For each $i \varepsilon T_{b-1}$, $S_i = S_k$ and therefore $E(S_i) \cap E(G_k) = \phi$. Since $E(G_k) \cap (\{e_i\} \cup E(S_i)) \neq \phi$, $e_i \varepsilon E(G_k)$ for all $i \varepsilon T_{b-1}$. This contradicts the fact that G_k has n edges and completes the proof.

Examples

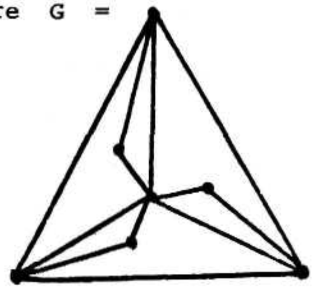
In general it is difficult to determine the graphs in $\mathcal{R}(G,H)$, even if the pair (G,H) is Ramsey-finite. In fact, the problem appears to be very difficult for $\mathcal{R}(mK_2,H)$. There is one trivial case and some small order cases that are known. For example one can verify directly the following

$$\mathcal{R}(K_2,H) = \{H\}.$$

$$\mathcal{R}(2K_2,2K_2) = \{C_5,3K_2\} \qquad [4]$$

$$\mathcal{R}(2K_2,K_3) = \{K_5,2K_3\ ,G\}$$

where G =



Although for a fixed m and a fixed graph H, $\mathcal{R}(mK_2,H)$ is finite, the cardinality of the set $\mathcal{R}(mK_2,H)$ is not bounded as m and H vary. For H either a dense graph or a very sparse graph this can be exhibited. First for $n \geq 4$ a family of $[(n+1)/2]$ non-isomorphic graphs in $\mathcal{R}(2K_2,K_n)$ will be described. Next for $n \geq 3$ a collection of $n - 2$ non-isomorphic graphs in $\mathcal{R}(2K_2,K_{1,n})$ will be described.

Let K be a graph isomorphic to K_{n+1}. Let $V(K) = R \cup S$ be partition of the vertices of K, and denote the cardinality of R by r. To each edge $e = xy$ with $\{x,y\} \subseteq R$ or $\{x,y\}$ S, associate a vertex v_e not in K and let v_e be adjacent to each vertex of K except for x and y. For $1 \leq r \leq [(n+1)/2]$, denote this graph by F_r.

$$|V(F_r)| = n + 1 + \binom{r}{2} + \binom{n+1-r}{2}$$

and

$$|E(F_r)| = \binom{n+1}{2} + (n-1)\left(\binom{r}{2} + \binom{n+1-r}{2}\right).$$

Clearly F_r is not isomorphic to $F_{r'}$ for $1 \leq r \neq r' \leq [(n+1)/2]$. It will be shown that $F_r \in \mathcal{R}(2K_2,K_n)$. Color the edges of F_r with colors red and blue. If there is no red $2K_2$, then the red subgraph is either a star or a triangle. It is easily checked for any vertex v of F_r or any triangle T contained in F_r that $F_r - v$ and $F_r - T$ contains a complete graph on n vertices. Therefore $F_r \to (2K_2,K_n)$. Let f be an arbitrary but fixed edge of F_r and consider the graph $F_r - f$. There exists an edge e of $K \subsetneq F_r$ such that f is in the unique complete graph on n vertices which contains v_e. Select a triangle T of $K \subsetneq F_r$ which contains e but does not have all of its vertices in the same term of the partition. One can check directly that $F_r - f - T$ does not contain a subgraph isomorphic to K_n. This implies $F_r - f \not\to (2K_2,K_n)$ and proves $F_r \in \mathcal{R}(2K_2,K_n)$.

For $3 \leq t \leq n$, let L be a completely disconnected graph on t vertices. For each vertex w of L associate distinct vertices $v(w), v_1(w), \ldots, v_{n-t+1}(w)$ not in L. Let $v(w)$ be adjacent to precisely the vertices

$$v_1(w), v_2(w), \ldots, v_{n-t+1}(w)$$

and the vertices of L except for w. Denote this graph by F_t. Thus F_t has $t + t(n-t+2)$ vertices, tn edges, and is the union (not vertex disjoint) of t stars with n edges. One can check directly that $F_t \in \mathcal{R}(2K_2, K_{1,n})$ for $3 \leq t \leq n$.

Conjectures

For an arbitrary graph H, the pair (mK_2, H) is Ramsey-finite. In [3] it was proved that if H is a 2-connected graph then the pairs $(K_{1,2}, H)$ and $(K_{1,3}, H)$ are Ramsey-infinite. This leads one to the following conjecture.

Conjecture 1: If G is a graph such that for any graph H, the pair (G,H) is Ramsey-finite, then G is isomorphic to mK_2 for some positive integer m.

A more difficult problem is to determine which pairs (G,H) are Ramsey-finite. Known results give support to the following conjecture.

Conjecture 2: The pair (G,H) is Ramsey-finite if and only if either

1) G or H is isomorphic to mK_2 or
2) G and H are both forests of stars with an odd number of edges.

References

[1] C. Berge, Sur le Couplage Maximum d'un Graphe, C. R. Acad. Sciences, Paris 247, 1958, 258-259.

[2] S. A. Burr, A Survey of Noncomplete Ramsey Theory for Graphs, to appear in the Proc. of the Nat. Acad. of Sciences of New York.

[3] S. A. Burr, P. Erdös, R. J. Faudree and R. H. Schelp, A Class of Ramsey-Infinite Graphs, to appear.

[4] S. A. Burr, P. Erdös and L. Loväsz, On Graphs of Ramsey Type, Ars Combinatoria, 1, 1976, 167-190.

[5] F. Harary, Graph Theory, Addison-Wesley, Reading Mass.,1969.

[6] J. Nésétril and V. Rődl, The Structure of Critical Ramsey Graphs.