1977.27 - (76.38) Problems and results on combinatorial number theory, III., *Number theory day (Proc. Conf., Rockefeller Univ., New York, 1976), Lecture Notes in Math., 626*, pp. 43–72, Springer, Berlin, 1977; **MR** 57#12442; **Zbl.** 368.10002.

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Problems and results on combinatorial number theory III

Paul Erdős

Like the two previous papers of the same title (I will refer to them as I and II) I will discuss problems in number theory which have a combinatorial flavor. To avoid repetitions and to shorten the paper as much as possible I will refer to previous results whenever convenient and will state as many new problems as possible, and will discuss the old problems only when they were neglected or if some new result has been obtained.

P. Erdős, Problems and results on combinatorial number theory I and II, a survey of combinatorial theory, 1973, North Holland, 117-138; Journées Arithmétiques de Bordeaux Juin 1974, Astérisque Nos. 24-25, 295-310. Both of these papers have many references. See also Quelques problèmes de la théorie des nombres, Monographies de l' Enseignement Mathématique No. 6, Univ. de Geneva (1963), 81-135. Graham and I will soon publish a paper which brings this paper up to date.

P. Erdős, Some unsolved problems, Michigan Math. J. 4(1957), 291-300 and Publ. Math. Inst. Hungar. Acad. Sci. 6(1961), 221-254.

I. First I discuss Van der Waerden' s and Szemerédi' s theorem and related questions. Denote by $f(n)$ the smallest integer so that if we divide the integers not exceeding n into two classes then as least one of them contains an arithmetic progression of n terms. More generally, denote by $f_u(n)$ the largest integer so that we can divide the integers not exceeding $f_u(n)$ into two classes so that in every arithmetic progression of n terms each class has fewer than $\frac{n+u}{2}$ terms.

The best lower bound for $f(n)$ is due to Berlekamp, Lovász and myself, ($f(p) > p^{2^p}$ if p is a prime and $f(n) > c2^n$ for all n). It would be very interesting to decide if $f(n)^{1/n} \to \infty$ is true. My guess would be that it is true. I proved by the probabilistic method that $f_u(n) > (1+\epsilon_c)^n$ if $u > cn$. The proof gives nothing if u is $0(n^{1/2})$. It would be very interesting to give some usable upper and lower bounds for $f_u(n)$. As far as I know the only result is due to J. Spencer who proved (Bull. Canad. Math. Soc. 16(1973), 464) $f_1(n) = n(n-1)$, equality only if $n = 2^t$. $f_2(n)$ is not known.

For various other generalizations (see II).

Denote by $r_k(n)$ the smallest integer so that every sequence $1 \leq a_1 < \dots < a_\ell \leq n$, $\ell = r_k(n)$ contains an arithmetic progression of k terms.

Szemerédi recently proved the old conjecture of Turán and myself

(1) $$r_k(n) = 0(n).$$

(1) of course contains Van der Waerden's theorem. The best estimates for $r_3(n)$ are due to Behrend and Roth who proved

$$\frac{n}{e^{c_1(\log n)^{1/2}}} < r_3(n) < \frac{c_2 n}{\log\log n} .$$

The true order of magnitude of $r_k(n)$ is very difficult to determine. I would expect that

$$r_k(n) = 0\left(\frac{n}{(\log n)^{\ell}}\right)$$

holds for every k and ℓ. A very attractive conjecture of mine states: Let $1 \leq a_1 < a_2 < \dots$ be an arbitrary sequence of integers with $\Sigma \frac{1}{a_i} = \infty$. Then our sequence contains for every k an arithmetic progression of k terms. I offer 3000 dollars for a proof or disproof of this conjecture. The conjecture, if true, would imply that for every k there are k primes in arithmetic progression.

There is an interesting finite form of our conjecture: Put

$$A_k = \max \sum_i \frac{1}{a_i}$$

where the maximum is extended over all sequences $a_1 < a_2 < \dots$ which do not contain an arithmetic progression of k terms. It is not at all obvious that $A_k < \infty$, in fact perhaps already $A_3 = \infty$, it would be very desirable to have good upper and lower bounds for A_k. I am afraid upper bounds are hopeless at present - so one should perhaps concentrate on lower bounds. I observed that $A_k > \frac{k \log 2}{2}$ and Gerver recently proved

(2) $$A_k > (1+\sigma(1))k \log k .$$

His proof will soon appear in Proc. Amer. Math. Soc. Gerver believes that (2) may be best possible.

Perhaps the following two further functions are of some interest: Put

$$A_k(n) = \max \sum_{a_i < n} \frac{1}{a_i}$$

where the maximum is to be taken over all sequences which do not contain an arithmetic progression of k terms. I of course expect $A_k(n) < c_k$. It might be of interest to estimate from above and below $A_k - A_k(n)$. Define next

$$A_k^{(n)} = \max \sum_{a_i > n} \frac{1}{a_i}$$

where the a_i do not contain an arithmetic progression of k terms. I expect that for every $k \lim_{n=\infty} A_k^{(n)} = 0$. It might be of interest to investigate $A_k - A_k^{(n)}$ for fixed n and large k. In particular is it $0(\log n)$? Also what happens if both k and n tend to infinity? It is not clear which if any of these questions will lead to fruitful results.

A sequence of integers $1 \leq a_1 < \ldots < a_k \leq n$ is called non-averaging if no a_i is the arithmetic mean of other a's.

The study of these sequences was started by E. Straus. Put $\max k = g(n)$. We have

$$e^{c(\log n)^{1/2}} < g(n) < n^{2/3+\varepsilon} . \tag{3}$$

The lower bound in (3) is due to Straus the upper bound to Straus and myself. Abbott recently proved the unexpected $g(n) > cn^{1/10}$. It would be very interesting to determine $\lim_{n=\infty} \log g(n)/\log n$.

Very recently Furstenberg proved Szemerédi's theorem by methods of ergodic theory, his proof will be published soon.

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2. Covering congruences and related questions.

A system of congruences

$$a_i \pmod{n_i}, \qquad 1<n_1<\ldots<n_k \tag{1}$$

is called a covering system if every integer satisfies at least one of the congruences (1). The principal conjecture which is now more than 40 years old states that n_1 can be arbitrarily large. It is surprising how difficult this conjecture is -- I offer 500 dollars for a proof or disproof. The record is still held by Choi who gives a system with $n_1 = 20$. Put

$$u_n = \min \sum_{n_1=n} \frac{1}{n_i}$$

where the minimum is to be taken over all covering systems $a_i \pmod{n_i}$. I conjecture that

$$u_n \to \infty \text{ as } n \to \infty. \tag{2}$$

If (2) is true it would be interesting to estimate u_n from above and below. Put

$$f(n) = \min k \text{ and } F(n) = \min n_k$$

where the minimum is extended over all systems (1) with $n_1 = n$. It would be very interesting to get non-trivial bounds for $f(n)$ and $F(n)$.

Here perhaps it is worthwhile to introduce a new parameter. Put

$$u_n(c) = \min \sum \frac{1}{n_i}, \qquad n_1 = n$$

where the minimum is extended over all finite systems $a_i \pmod{n_i}$, $n = n_1 < n_2 < \ldots$ for which the density of integers not satisfying any of these congruences is less than or equal to c. Estimate or determine the asymptotic properties of $u_n(c)$ as $c \to 0$ and $n \to \infty$. Similar questions can be asked about $f(n, c)$ and $F(n, c)$.

I conjecture that for every ℓ there is a covering system (1) where all the n_i are square-free integers all whose prime factors are greater than p_ℓ.

Let $n_1 < n_2 < \ldots < n_k$ be a sequence of moduli. It would be very interesting to obtain conditions (if possible necessary and sufficient ones) that a covering system (1) exists. In particular I conjecture that for every C there is an n with $\sigma(n)/n > C$, but no system (1) exists where the $n_i > 1$ are the divisors of n. On the other hand Benkoski and I conjectured that if $\sigma(n)/n > C$ then n is the sum of distinct proper divisors of n. If this conjecture is true we want to estimate the smallest value of C for which the conjecture holds.

An older conjecture of Benkoski states: if n is odd and $\frac{\sigma(n)}{n} \geq 2$ then n is the sum of distinct proper divisors of n.

One can also study infinite covering systems as was done by Selfridge and his students but to avoid trivialities one usually insists: every $m > m_0$ must satisfy a congruence $m \equiv a_i \pmod{n_i}$, $m \geq n_i$. Another possibility would be to require that if $k > k_0(\varepsilon)$ the density of the integers satisfying none of the congruences $a_i \pmod{n_i}$ $1 \leq i \leq k$ is less than ε. Perhaps the first condition implies the second.

Denote by N the sequence $1 < n_1 < n_2 < \ldots$ of moduli. N has property P_1 if for every choice of residues $a_i \pmod{n_i}$ and to every $\varepsilon > 0$ there is a k so that the density of integers satisfying none of the congruences

$$(3) \qquad a_i \pmod{n_i} \qquad 1 \leq i \leq k$$

is less than ε. N is said to have property P_2 if there is a sequence of residues a_i so that the density of the integers satisfying none of the congruences (3) is less than ε. It has property P_3 if this holds for almost all (i.e. $0(\prod_{i=1}^{k} n_i)$) choices of the residues a_i. P_3 clearly holds if there is a subsequence $\{n_{i_r}\}$ with $\sum_r \frac{1}{n_{i_r}} = \infty$, $(n_{i_{r_1}}, n_{i_{r_2}}) = 1$, but at the moment I do not see a necessary and sufficient condition. P_2 certainly holds if $\sum_{i=1} \frac{1}{n_i} = \infty$, but it also holds if $n_i = 2^i$. P_1 no doubt holds if and only if $\sum_i \frac{1}{n_i} = \infty$. I formulated these problems while writing these lines and must ask the indulgence of the reader if some of the questions are trivial or false.

P_1 is clearly equivalent with the condition: For every choice of the residues a_i the density of integers which does not satisfy any of the congruences $a_i \pmod{n_i}$, $1 \leq k < \infty$ is 0. On the other hand observe that it is trivial that one can find residues a_i so that every integer satisfies at least one of the congruences $a_i \pmod{n_i}$ -- suffices to choose $a_i = i$. By a slight modification we can obtain a problem which is perhaps not trivial: Let $n_1 < n_2 < \dots$ what is the necessary and sufficient condition that residues a_i exist so that all but a finite number of integers m satisfy one of the congruences

$$m \equiv a_i (\mathrm{mod}\ n_i), \qquad m > n_i . \tag{4}$$

One can also ask: What is the necessary and sufficient condition that almost all integers satisfy one of the congruences (4)?

For particular choices of the a_i (say $a_i = 0$) it often is very hard to decide if almost all integers satisfy one of the congruences $a_i(\mathrm{mod}\ n_i)$. A very old problem of mine states: Is it true that almost all integers have two divisors $d_1 < d_2 < 2d_1$.

If this conjecture is correct one could choose as moduli the integers which are minimal relative to the property of having two divisors d_1, d_2 with $d_1 < d_2 < 2d_1$ in the sense that no proper divisor has that property. The choice $a_i = 0$ would then determine a set satisfying at least one of the congruences with infinite complement and density 1.

Many further questions can be asked but I leave their formulation to the reader.

A set of congruences $a_i(\mathrm{mod}\ n_i)$, $n_1 < n_2 < \dots$ is called disjoint if every integer satisfies at most one of these congruences. I conjectured that no covering system can be exact i. e. every integer satisfies exactly one of the covering congruences. Mirsky and Newman and a little later Davenport and Rado found a very simple proof of my conjecture.

Stein and I asked: Let

$$a_i(\mathrm{mod}\ n_i), \qquad 1 < n_1 < \dots < n_k \leq x \tag{5}$$

be a disjoint system. Put $\max k = g(x)$, determine or estimate $g(x)$ as accurately as possible. Szemeredi and I proved

$$x e^{-c_1 (\log x)^{\frac{1}{2}+\epsilon}} < g(x) < \frac{x}{(\log x)^{c_2}} .$$

We believe that the lower bound is closer to the truth. Szemerédi and I tried unsuccessfully to give necessary and sufficient conditions for a sequence of moduli $n_1 < \ldots < n_k$ that a disjoint system $a_i \pmod{n_i}$, $1 \leq i \leq k$ should exist.

As far as I know the following question which may be of some interest has not yet been investigated: Let all the n_i for which (5) is a disjoint system be greater than m. $\varepsilon_m = \max \Sigma \frac{1}{n_i}$. Probably $\varepsilon_m \to 0$ as $m \to \infty$. If true estimate ε_m. Perhaps it would be better to require that all prime factors of the n_i are greater than m.

There are many recent generalisations of covering congruences and exact covering congruences. Here I only state a beautiful conjecture of Herzog and Schönheim: Let ζ be a finite Abelian group. $H_1, \ldots, H_k$ are cosets of different sizes. Prove that $\bigcup_{i=1}^{k} H_i$ never gives an exact covering of ζ.

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3. Some applications of covering congruences

In 1934 Romanoff proved that the lower density of integers of the form $2^k + p$ is positive. He wrote me whether I can prove that there are infinitely many odd integers not of the form $2^k + p$. This question led me to the problems on covering congruences and I proved -- using covering congruences -- that there is an arithmetic progression consisting entirely of odd numbers no term of which is of the form $2^k + p$ (this was proved independently by Van der Corput too).

It is easy to see that if one could prove that there are covering systems with arbitrarily large n_1 then it would follow that for every r there is an arithmetic progression no term of which is of the form $2^k + \theta_r$ where θ_r has at most r distinct prime factors.

The following question seems very difficult: Is it true that for every r there are infinitely many odd integers not the sum of a prime and r or fewer powers of 2? Is the density of these integers positive? Do they contain an infinite arithmetic progression? It is extremely doubtful if covering congruences will help here. Schinzel proved that there are infinitely many odd integers not of the form $p + 2^k + 2^\ell$. In the opposite direction Gallagher recently proved (using the method of Linnik) that to every $\varepsilon > 0$ there is an $r > r_0(\varepsilon)$ so that the lower density of integers of the form $p + 2^{k_1} + \ldots + 2^{k_r}$ is greater than $1 - \varepsilon$.

Is it true that every sufficiently large odd integer is of the form $2^k + \theta$ where θ is squarefree? Is there in fact an odd integer not of this form?

The connection with covering congruences becomes apparent if we pose the following question: Let $p_1, \ldots, p_k$ be any finite set of primes. Is it true that every sufficiently large odd integer is of the form $2^k + L$ where $p_i^2 \nmid L$ for every $i = 1, \ldots, k$?

Denote by $f(n)$ the number of solutions of $2^k + p = n$ and let $a_1 < a_2 < \ldots$ be the sequence of integers with $f(n) > 0$. In view of Romanoff's result one would hope that the density of our sequence $\{a_i\}$ exists. Unfortunately to decide questions of this type seems to be far beyond our resources. I proved that for infinitely many $n, f(n) > c \log \log n$ but could not decide whether $f(n) = 0(\log n)$. I conjectured that 105 is the largest integer for which all the numbers $n - 2^k$, $1 \leq k < \frac{\log n}{\log 2}$ are primes. I am fairly certain that this conjecture is true. On the other hand it seems likely that for infinitely many n all the integers $n - 2^k$, $2^k < n$ are squarefree.

Incidentally I am sure that $\overline{\lim}\,(a_{i+1} - a_i) = \infty$. This would certainly follow if there are covering systems with arbitrarily large n_1.

The following somewhat vague conjecture can be formulated. Consider all the arithmetic progressions (of odd numbers) no term of which is of the form $2^k + p$. Is it true that all these progressions can be obtained from covering congruences and that all (perhaps with a finite number of exceptions) integers not in any of these progressions are of the form $2^k + p$?

Finally Cohen and Selfridge proved by covering congruences that there is an arithmetic progression of odd numbers no term of which is of the form $2^k \pm p^\alpha$ and Schinzel used covering congruences for the study of irreducibility of polynomials.

References

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4. Some unconventional extremal problems

An infinite sequence $1 \leq a_1 < \ldots$ of integers is called an A sequence if no a_i is the distinct sum of other a's. I proved that for every A sequence $\Sigma \frac{1}{a_i} < 100$. Sullivan obtained a very substantial improvement, he proved $\Sigma \frac{1}{a_i} < 4$. It would be interesting to determine $\max \Sigma \frac{1}{a_i}$ where the maximum is extended over all A sequences. Sullivan conjectures that this maximum is only a little greater than 2. Is it possible to obtain necessary and sufficient conditions for $b_1 < b_2 < \ldots$ so that there should be an A sequence satisfying $a_n < cb_n$ for some absolute constant c and every n? The same question can be asked for all the other sequences considered in this paper. I refer to this problem as (I).

Perhaps the inequalities of Levine and Sullivan can solve problem (I) for A sequences (see their forthcoming paper in Acta Arithmetica).

I conjectured and Levine just proved that if $n \leq a_1 < \ldots$ is an A sequence then $\Sigma \frac{1}{a_i} < \log 2 + \varepsilon_n$ where $\varepsilon_n \to 0$ as n tends to infinity, $n, n+1, \ldots, 2n$ shows that this is best possible.

Usually the exact determination of these extremal problems is difficult and one is rarely successful. I conjectured and Ryavec and others proved that if $1 \leq a_i < \ldots < a_n$ is a sequence of integers such that all the sums $\sum_{i=1}^{n} \varepsilon_i a_i, \varepsilon_i = 0$ or 1 are all distinct then $\sum_{i=1}^{n} \frac{1}{a_i} \leq 2 - \frac{1}{2^{n-1}}$ equality if and only if $a_i = 2^{i-1}$.

Here I call attention to one of my oldest problems (which is of course mentioned in I and II): Let $1 \leq a_1 < \ldots < a_n \leq x$ be such that all the sums $\sum_{i=1}^{n} \varepsilon_i a_i$ are distinct. Is it true that $n < \frac{\log x}{\log 2} + C$? I offer 300 dollars for a proof or disproof. As far as I know this could hold with $C = 3$.

Another couple of simple extremal problems which I can not solve state as follows: Let $1 \leq a_1 < \ldots$ be a sequence of integers for which all the sums $a_i + a_j$ are different. Determine $\max \Sigma \frac{1}{a_i}$. We get different problems if $i = j$ is permited or not -- but I can not solve any of them.

Let $a_0 = 0$, $a_1 = 1 < a_2 < \ldots$ be such that every integer is the sum of two a's. Determine $\min \Sigma \frac{1}{a_i}$.

In some cases one encounters problems where it is not hard to prove that our sequence has density 0 but it is much harder to prove that $\sum_i \frac{1}{a_i} < \infty$, e.g. let $a_1 < a_2 < \ldots$ be an infinite sequence of integers where no a_i divides the sum of two greater a's. Sárkozi and I proved that the density of such a sequence is 0 but we could not prove $\Sigma \frac{1}{a_i} < \infty$ and are nowhere near of settling problem (I). The following finite problem remains here. Let $1 \leq a_1 < \ldots < a_k \leq x$ be such that no a_i divides the sum of two greater a's. Then $k \leq [\frac{x}{3}] + 1$. Equality, say, if $x = 3n$ and the a's are the integers $2n, 2n+1, \ldots, 3n$.

References

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5. Some more extremal problems in additive and multiplicative number theory

Sidon calls a sequence of integers $1 \leq a_1 < \dots$ a B_k sequence if the sums $\sum_{i=1}^{k} \varepsilon_i a_{r_i}, \varepsilon_i = 0$ or 1 are all distinct. Sidon asked in 1933: find B_k sequence for which a_n tends to infinity as slowly as possible. It is easy to see that there is a B_2 sequence with $a_n < C n^3$ for all n. One of the most challenging problems here states: Is there a B_2 sequence with $a_n/n^3 \to 0$ -- I give 100 dollars for a proof or disproof. I of course expect that such a sequence exists -- in fact I am sure that there is a B_2 sequence with $a_n < n^{2+\varepsilon}$ for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$. Rényi and I proved by probabilistic methods that there is a sequence $a_n < n^{2+\varepsilon}$ for which $f(m) = \sum_{a_i+a_j=m} 1 < c_\varepsilon$ where the constant c_ε depends only on ε. On the other hand I proved that for every B_2 sequence

$$(1) \qquad \limsup \frac{a_n}{n^2} = \infty \text{ in fact } \limsup_{n=\infty} \frac{a_n}{n^2 \log n} > 0.$$

Can (1) be improved? An old conjecture of Turán and myself states that if $\{a_n\}$ is a basis then $\limsup f(n) = \infty$, more generally: let $a_n < c\, n^2$, $n = 1, 2, \dots$ is it then true that $\limsup f(n) = \infty$? I offered and offer 300 dollars for a proof or disproof of these conjectures.

The B sequences behave quite differently if we restrict them to a finite interval. Denote by $B_k(n)$ the maximum number of terms of a B_k sequence not exceeding n. Turán and I (see also Chowla) proved

$$(1) \qquad (1+o(1))n^{1/2} < B_2(n) < n^{1/2} + cn^{1/4}.$$

(1) is of course mentioned in I and II. We conjecture

$$(2) \qquad B_2(n) = n^{1/2} + o(1).$$

I offer 300 dollars for a proof or disproof of (2).

Bose and Chowla observed that it is very hard to estimate $B_k(n)$ from above for $k \geq 3$. They observe $B_3(n) \geq (1+o(1))n^{1/3}$ but remark that our proof with Turan breaks down and $B_3(n) \leq (1+o(1))n^{1/3}$ is open.

Incidentally if $a_1 < a_2 < \dots$ is an infinite B_3 sequence I cannot prove that $\limsup a_n/n^3 = \infty$, though I have no doubt that this is true.

We are very far from being able to solve problem (I) for B_k sequences -- as far as I know there is not even a reasonable conjecture.

Levine in a recent letter to me asked: Is there a B_2 sequence and an $\varepsilon > 0$ which $\sum_i \frac{1}{a_i^{1/2}(\log a_i)^\varepsilon} = \infty$? It follows from my results that for every $\delta > 1$

(3) $$\sum_i \frac{1}{(a_i \log a_i)^{1/2}(\log \log a_i)^\delta} < \infty .$$

But I have an example of a B_2 sequence for which

(4) $$\sum_i \frac{1}{a_i^{1/2}(\log \log a_i)} = \infty .$$

In trying to close the gap between (3) and (4) Levine asked: Is it true that

$$\sum \frac{1}{(a_i \log a_i)^{1/2}(\log \log a_i)}$$

converges for every B_2 sequence?

I proved: There exists an infinite B_2 sequence with

(5) $$\limsup_{n=\infty} \frac{B_2(n)}{n^{1/2}} \geq \frac{1}{2} ,$$

$1/2$ in (5) was improved to $1/2^{1/2}$ by Kruckeberg. In view of (1) the best possible result could be 1. This would follow if the following conjecture of mine would hold: Let $a_1 < a_2 < \dots < a_k$ be any B_2 sequence. Then there exists a modulus m and a perfect difference set mod m which contains the a's. In other words: There is an integer $m = u^2 + u + 1$ and $u + 1$ residues mod m, $b_1, \dots, b_{u+1}$ so that every residue mod m is uniquely of the form $b_i - b_j$ and the a's occur amongst the b's.

This conjecture if true would seem to me to be very interesting. Questions like this have been investigated and in some cases solved by Treash, Lindner and others, for Steiner systems and other more complicated combinatorial structures.

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6. Problems on infinite subsets

Graham and Rothschild conjectured that if we split the integers into two classes then there always is an infinite sequence $a_1 < a_2 < \dots$ so that all the finite sums

$$(1) \qquad \Sigma \varepsilon_k a_k, \quad \varepsilon_k = 0 \text{ or } 1.$$

are in the same class.

This conjecture was proved recently by Hindman and the proof was simplified by Baumgartner. I just heard that Glaser using an idea of Galvin obtained a very interesting topological proof of the theorem.

A few days ago I asked: Is there a function $f(n)$ so that if we split the integers into two classes there always is a sequence $a_1 < \dots$ for which $a_n < f(n)$ holds for infinitely many n and so that (1) holds?

Galvin just showed that no such $f(n)$ exists. To see this he defines a splitting as follows: Let $F(m) \to \infty$ sufficiently fast. Put $n = 2^x y$, y odd. n is in the first class if $y \geq F(x)$ and is in the second class if $y < F(x)$. It is easy to see that this construction gives a counter example.

There might be two ways to save the situation and obtain some nontrivial problem. Is it true that there is an $f(n)$ so that if we split the integers into k (or $\aleph_0$) classes there is a sequence $a_1 < a_2 < \dots, a_n < f(n)$ so that at least one of the classes is disjoint from the set of all sums $\Sigma \varepsilon_k a_k$? One would even ask a weaker statement: Divide the integers into continuum many almost disjoint classes, i.e. $\{A_\alpha\}$ is a set of integers $1 \leq \alpha < \omega_c$ $(A_{\alpha_1} \cap A_{\alpha_2}) < \aleph_0$ and ω_c is the initial ordinal of the continuum, then there is an infinite sequence $\{a_n\}$, $a_n < f(n)$, for infinitely many n and the sums (1) do not meet all the classes A_α?

The second possibility would be: Split the real numbers into two classes. Is there a sequence $\{x_n\} x_n < f(n)$ for infinitely many n so that all the sums $\Sigma \varepsilon_k x_k$ are in the same class? Is the following true: Let S_x be a set of real numbers so that the equation $x + y = z$ is not solvable in S. Is there then a set $\{x_\alpha\}$ of power c in the complement of S so that all the sums $\{x_{\alpha_1} + x_{\alpha_2}\}$ also belong to the complement of S? If the answer is no then we could perhaps assume that all the $x + y$, $x \in S$, $y \in S$ are distinct.

I thought of strengthening Hindman's theorem in the same way as Szemerédi strengthened Van der Waerden's theorem. Is the following true: Let A be a sequence of positive density. Is there an infinite sequence $a_1 < a_2 < \dots$ and an integer t so that all the integers $a_i + a_j + t$ are in the same class? Straus observed that the full strength of Hindman's theorem does not hold in this case.

Some time ago I thought of the following fascinating possibility: Divide the integers into two classes. Is it true that there always is a sequence $a_1, a_2, \ldots$ so that all the finite sums $\Sigma \varepsilon_i a_i$ and all the finite products $\prod_i a_i^{\varepsilon_i}$ are in the same class. At this moment the problem is open. More generally one can ask: Is there an infinite sequence $a_1 < a_2 < \ldots$ so that all the multilinear expressions formed from the a's are in the same class? One would perhaps guess that the answer is no but no counter example is in sight.

The following much weaker conjecture is also open: Is there a sequence $a_1 < a_2 < \ldots$ so that all the sums $a_i + a_j$ and products $a_i a_j$ are in the same class? Perhaps we should also require that the a_i are also in the same class. Graham proved that if we divide the integers ≤ 252 into two classes there are four distinct numbers $x, y, x+y, xy$ all in the same class - 252 is best possible. Hindman proved that if we divide the integers $2 \leq t \leq 990$ into two classes, then there are always four distinct numbers all greater than 1 $x, y, x+y, xy$ all of the same class. So far nothing is known in case we assume all the integers ≥ 3.

Answering a question of Ewings, Hindman proved (will appea r soon in the Journal of combinatorial theory), that if we divide the integers into two classes there always is an infinite sequence $x_1 < x_2 < \ldots$ so that all the sums $x_i + x_j$ ($i=j$ permitted) are in the same class.

Hindman just informed me that there may be a gap in his proof, but I hope that this will be corrected by the time this paper appears. On the other hand he found a decomposition into three classes A_1, A_2, A_3 so that no such infinite sequence exists. In fact Hindman observes that one of his sequences say A_1 has density 0. In his example

$$A_1(x) = \sum_{\substack{a_i \in A_1 \\ a_i \leq x}} 1 < cx^{1/2}. \tag{1}$$

It is not yet clear whether (and to what extent) (1) can be improved.

More than 10 years ago R. L. Graham and I conjectured that if we split the integers into two classes then

$$1 = \sum_1 \frac{1}{x_i}, \qquad x_1 < x_2 < \ldots \qquad \text{(finite sum)} \tag{2}$$

is always solvable with the x_i all in the same class. This should probably not be too difficult. Clearly many generalisations are possible.

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7. A new extremal problem

Let $1 \leq a_1 < \ldots < a_n$ be a sequence of integers. Denote by $f_r(n)$ the minimum number of distinct integers which are the sum or product of exactly r of the a' s.

I conjectured that for every r and $\varepsilon > 0$ if $n > n_0(\varepsilon, r)$, $f_r(n) > n^{r-\varepsilon}$.

If true this seems extremely difficult and seems to require new ideas (unless of course an obvious point is being overlooked).

Szemerédi and I observed that it follows from deep results of Freiman that

$$\lim_{n=\infty} f_2(n)/n = \infty ,$$

but even the proof of $f_2(n) > n^{1+\varepsilon}$ seems to present great difficulties.

$$f_2(n) > \frac{n^2}{(\log n)^k}$$

is certainly false. Perhaps the true order of magnitude of $f_2(n)$ is

$$n^2 \exp(-c \log n/\log \log n) .$$

Denote by $F(n)$ the smallest integer so that there are at least $F(n)$ distinct integers which are the sum or product of distinct a_i's. It seems certain that

$$F(n) > n^k \qquad \text{for } n > n_0(k)$$

but I have not been able to prove this. By a remark of E. Straus it holds for $k = 2$.

It is not hard to see that $F(n)^{1/n} \to 1$. Perhaps the true order of magnitude of $F(n)$ is

$$\text{(1)} \qquad \exp(\log n)^c , \qquad c > 1$$

perhaps (1) holds for every c.

The following more general problem might be of interest. Let $G(n;k)$ be a graph of n vertices and k edges. To each vertex of G we associate an integer $x_i, x_i \neq x_j$, $1 \leq i < j \leq n$. If x_i is joined to x_j we associate to the edge the two integers $x_i + x_j$ and $x_i x_j$. Thus we associate $2k$ integers to the graph

$G(n;k)$. Denote by $A(G(n;k))$ the smallest number of distinct integers corresponding to $G(n;k)$. Perhaps if $\frac{\log k}{\log n} \to 2$ then $A(G(n;k)) > n^{2-\varepsilon}$ for every $G > 0$ and $n > n_0$. This conjecture if true is a far reaching extension of my original conjecture $f_2(n) > n^{2-\varepsilon}$.

All these conjectures can be extended to the case when the x_i are real or complex numbers or elements of a vector space.

For a few weeks I thought that the following result might hold (here $n = 2m$ and our graph is regular of degree one). Let $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ be $2n$ integers. Then there are at least $n+1$ (or at least cn) distinct numbers among the $2n$ numbers $\{a_i+b_i, a_ib_i\}$, $i = 1, 2, \ldots, n$. A. Rubin showed that I was much too optimistic. The conjecture certainly fails for $c > 1/2$ and if the a_i's and b's can be real numbers then there do not have to be more than $cn^{1/2}$ distinct numbers amongst the $\{a_i+b_i, a_ib_i\}$. It is almost certain that the same holds if the a_i and b_i are restricted to be integers, but as far as I know Rubin did not yet work out the details. If we assume $k > n^{1+\varepsilon}$ or perhaps only $k/n \to \infty$ one perhaps might get some results but I do not have any plausible conjecture so far.

8. Some unconventional problems on primes

Is there a sequence $a_1 < a_2 < \dots$ of integers satisfying $A(x) = \sum_{a_i < x} 1 < \log x$ so that all sufficiently large integers are of the form $p + a_i$? If this is impossible then perhaps such a sequence exists for which the density of integers not of the form $p + a_i$ is 0. Clearly many similar questions can be asked for other sequences then the primes but there are very few results. Ruzsa proved that there is a sequence of integers $a_1 < a_2 < \dots$, $A(x) < cx/\log x$ so that every integer is of the form $2^k + a_i$. Is it true that there is such a sequence for every $c < \log 2 + \varepsilon$?

The prime k-tuple conjecture states: Let $a_1 < \dots < a_\ell$ be a set of integers which do not form a complete set of residues mod p for any p. Then there are infinitely many integers n so that all the integers $\{n+a_i\}$ $1 \le i \le k$ are primes. This problem is unattackable at present. Let $a_1 < a_2 < \dots$ be an infinite sequence of integers. It would be interesting to find a necessary and sufficient condition for the existence of infinitely many n so that all the integers $n + a_i$ are prime. Perhaps it would be more reasonable to permit for each n a finite number of exceptions. An old and very fascinating conjecture of Ostman should be mentioned here: Are there two sequences $a_1 < a_2 < \dots$; $b_1 < b_2 < \dots$ so that all but a finite number of the sums $a_i + b_j$ are primes and all but a finite number of primes are of the form $a_i + b_j$? The answer is obviously No! but unfortunately nobody can prove this. Hornfeck showed that both sequences must be infinite.

Let us now return to our problem. If n is such that all the integers $n + a_i$, $i = 1, 2, \dots$ are primes we first of all clearly must have $(n+a_i) \nmid (n+a_j)$ and what is more $(n+a_i, n+a_j) = 1$. Is it possible to find a necessary and sufficient condition for the following three properties of an infinite sequence A? There is another infinite sequence B so that 1. $(a_i+b_j) \nmid (a_r+b_s)$, 2. $(a_i+b_j, a_r+b_s) = 1$, 3. $a_i + b_j$ are all primes? I think property 1 and 2 can probably be handled, but I had no time to think this over carefully. For 3 the only hope would be a reasonable conjecture. The problems may change if we permit for each a_i and b_j a finite number of exceptions also we could restrict ourselves to asking: assume $(a_i+b_{j_1}, a_i+b_{j_2}) = 1$ for $1 \le j_1 < j_2 < \infty$.

De Bruijn, Turán and I considered the function

(1) $$f(n) = \sum_{p < n} \frac{1}{n-p}.$$

It is not difficult to show that

$$(2) \qquad \frac{1}{x}\sum_{n<x} f(n) \to 1, \quad \frac{1}{x}\sum_{n<x} f^2(n) \to 1 .$$

It follows from Hoheisel's classical $\pi(n+n^{\alpha}) - \pi(n) > c\,\frac{n^{\alpha}}{\log n}$ that $\liminf_{n=\infty} f(n) > 0$. It is likely that

$$(3) \qquad \liminf_{n=\infty} f(n) = 1 , \qquad \limsup_{n=\infty} f(n) = \infty$$

Perhaps $f(n) = o(\log\log n)$. A weaker conjecture which is perhaps not quite inaccessible states: To every $\varepsilon > 0$ there is an x_0 so that for every $x > x_0$ there is a $y < x$ so that

$$(4) \qquad \pi(x) - \pi(y) < \varepsilon\, \pi(x-y) .$$

One in fact feels that $\pi(x) - \pi(y)$ should be usually of the order of magnitude $\frac{x-y}{\log x}$ and therefore it is reasonable to guess that (4) is satisfied for every $y < x - (\log x)^C$ for sufficiently large C. In fact I can not at this moment disprove:

$$(5) \qquad \pi(x) - \pi(y) < c_1 \frac{x-y}{\log x} \quad \text{for } y < x - (\log)^C$$

(5) would imply $f(n) < c \operatorname{logloglog} n$ and perhaps

$$\overline{\lim}\, f(n)/\operatorname{logloglog} n > 0.$$

We could try to study $f(p)$ but this is even harder than $f(n)$. I could not prove

$$\frac{1}{\pi(x)}\sum_{p<x} f^2(p) \to 1 .$$

I conjectured once optimistically that

$$(6) \qquad \Sigma_1 \frac{1}{p-p_j} = 1 + o(1)$$

where in $\Sigma_1\ p_j < p - \log p$. (6), if true, is of course hopeless.

Hensley and Richards recently showed that if the prime k-tuple conjecture is true (in fact it certainly "must" of course be true) then for every large y there are infinitely many x for which $\pi(x+y) > \pi(x) + \pi(y)$, and in fact for an absolute constant $c > 0$.

$$\pi(x) + \pi(y) + c\, y/_{(\log y)} < \pi(x+y)\,. \tag{7}$$

Richards and I have a forthcoming paper on some of these questions in Monatshefte der Mathematik. There is an important disagreement between us. Richards believes that (7) holds for arbitrarily large values of c and suitable x and y. I conjecture the opposite.

One final conjecture: Let $n < q_1 < \dots < q_k \leq m$ be the sequence of consecutive primes in (n, m)

$$\sum_{i=1}^{k} \frac{1}{q_i - n} < \sum_{p < m-n} \frac{1}{p} + c$$

for a certain absolute constant c. Trivially the opposite inequality is not true, since there are arbitrarily large gaps between the primes. It does not seem to be trivial to prove that

$$\liminf_{n=\infty} \left(\lim_{m=\infty} \sum_{n < q_i < m} \frac{1}{q_i - n} - \sum_{p<m} \frac{1}{p} \right) = -\infty \tag{8}$$

At present I do not see how to prove (8).

Eggleton, Selfridge and I are writing a long paper on somewhat unconventional problems in number theory. Our paper will appear in Utilitas Matematica. One of our problems related to (1) states as follows: Let $a_0 = 0$, $a_1 = 1$, a_k is the smallest integer for which $(n-a_k, n-a_i) = 1$ for all $0 \leq i < k$. Put

$$g(n) = \sum_{i=1} \frac{1}{a_i} \tag{9}$$

We conjecture $g(n) \to \infty$ as $n \to \infty$. This is probably very difficult. We can only prove $a_k < k^{2+\varepsilon}$ for $k > (\log n)^C$, $C = C(\varepsilon)$, but perhaps

$$a_k < C\, k \log k \text{ if } k > (\log k)^{\alpha_C} \tag{10}$$

where α_C depends on C. Perhaps (10) is a little too optimistic, but (10) certainly "must"(?) hold if $k > \exp(\log k)^{1/2}$ which would easily imply (9).

Straus and I conjectured: Let $p_1 < p_2 < \dots$ be the sequence of consecutive primes. Then for $k > k_0$ there always is an $i < k_0$ so that

(11) $$p_k^2 < p_{k+i}p_{k-i} .$$

Selfridge with whom we discussed this problem strongly doubted that (11) is true, in fact he expressed the opposite conjecture.

Denote by $f(k)$ the number of changes of signs of the sequence

$$p_k^2 - p_{k+i}p_{k-i}, \quad 0 < i < k .$$

Perhaps $f(k) \to \infty$ as k tends to infinity, this of course would be a very considerable strengthening of our conjecture with Straus. I cannot even prove $\lim_{k=\infty} \sup f(k) = \infty$. An old result of Turán and myself states that $p_k^2 - p_{k+1}p_{k-1}$ has infinitely many changes of signs.

Put $d_k = p_{k+1} - p_k$. Turán and I proved that $d_{k+1} > d_k$ and $d_{k+1} < d_k$ both have infinitely many solutions. We of course cannot prove that $d_k = d_{k+1}$ has infinitely many solutions. We further could not prove that $d_{k+2} > d_{k+1} > d_k$ has infinitely many solutions. It is particularly annoying that we could not prove that there is <u>no</u> k_0 so that for every $i > 0$.

(12) $d_{k_0+i} > d_{k_0+i+1}$ if $i \equiv 0 \pmod 2$ and $d_{k_0+i} < d_{k_0+i+1}$ if $i \equiv 1 \pmod 2$!!

Perhaps we overlooked a simple idea. Turán has some very challenging problems on consecutive primes: Is it true that for every d and infinitely many n $p_n \equiv p_{n+1} \pmod d$?

Finally, in connection of our conjecture with Straus and Selfridge's doubts, the following question of Selfridge and myself might be of interest: Let $a_1 < a_2 < \dots$ be a sequence of positive density. Is it true that for infinitely many k and every $1 \le i < k$

(13) $$a_k^2 > a_{k+i}a_{k-i} \ ?$$

Does (13) hold if the density of a's is 1?

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9. Some extremal problems in real and complex numbers

Many extremal problems on integers can be extended to real numbers. To explain what I have in mind consider the following problem: Let $1 \leq a_1 < \ldots < a_{k^{(n)}} \leq n$ be a sequence of integers. Assume that the products $a_i a_j$ are all distinct. Then $(0 < c_2 < c_1)$

$$(1) \qquad \pi(n) + c_2 n^{3/4} \Big/ (\log n)^{3/2} < \max k^{(n)} < \pi(n) + c_1 n^{3/4} \Big/ (\log n)^{3/2}$$

Probably there is a c so that

$$(2) \qquad \max k^{(n)} = \pi(n) + c\, n^{3/4} \Big/ (\log n)^{3/2} + o\left(n^{3/4} \Big/ (\log n)^{3/2}\right)$$

but (2) will not concern us now. Let $1 \leq a_1 < \ldots < a_{k^{(n)}} \leq n$ be a sequence of real numbers. Assume that

$$|a_u a_v - a_t a_s| \geq 1$$

for every choice of the indices u, v, t, p. Does (1) remain true? I cannot even prove $k^{(n)} = o(n)$.

Clearly nearly all the extremal problems in number theory which I considered during my long life can be extended in this way. In fact the a's could be complex numbers or more generally members of a vector space e.g. Let $h_1 \ldots h_k$, $|h_i| \leq n$ be n complex numbers assume that

$$|h_a h_b - h_c h_d| \geq 1$$

holds for every $1 \leq a, b, c, d \leq k$. How large can be $\max k$? $o(n^2)$ is really certain but at this moment I do not see it. If the h's are complex integers a result like (2) can undoubtedly be proved.

Now I discuss some more such questions. A sequence of integers $a_1 < a_2 < \ldots$ is called a primitive sequence if $a_i \nmid a_j$. Primitive sequences have been investigated a great deal see e.g. our survey paper with Sárközi and Szemerédi. But problem (I) is not yet solved for primitive sequences. As a first step to solve problem (I) one could characterize the sequences $n_1 < n_2 < \ldots$ for which there is an $\varepsilon > 0$ and a primitive sequence $a_1 < \ldots$ for which

$$A(2^{n_k}) > \varepsilon\, 2^{n_k}$$

for every $k = 1, 2, \ldots$.

The generalisations to real sequences seem to lead to interesting diophantine problems: Let $a_1 < a_2 < \ldots$ be a sequence of real numbers and assume that for every i, j, k

$$|ka_i - a_j| \geq 1 . \tag{3}$$

I cannot even prove that (3) implies

$$\underline{\lim}\, \frac{A(x)}{x} = 0, \quad \left(A(x) = \sum_{a_i < x} 1 \right) .$$

One would guess that most of the asymptotic properties which are valid for primitive sequences also hold if only (3) is assumed.

The only result is the following unpublished theorem of J. Haight. Assume that the a's are rationally independent and satisfy (3). Then $A(x)/x \to 0$. This in fact is not true for primitive sequences of integers. In view of Haight's result I believe that much of the difficulty will already be encountered if the a's are assumed to be rational numbers.

During my lecture at Queens College one member of the audience (perhaps S. Shapiro) asked the following question which I had overlooked: Let $1 < a_1 < \ldots$ be a sequence of real numbers. Assume that

$$\left| \prod_i a_i^{\alpha_i} - \prod_j a_j^{\beta_j} \right| \geq 1$$

for every pair of distinct choices of the finitely many non-negative integers α_i and β_j. Is it then true that

$$\sum_{a_i \leq x} 1 = A(x) \leq \pi(x) \ ? \tag{1}$$

(1) is certainly a fascinating conjecture. The a's are sometimes called Beurling prime numbers and have a large literature - as far as I know (1) has never been considered before. A very nice and unpublished conjecture of Beurling

states: Assume that the number of numbers of the form $\Pi a_i^{\alpha_i}$ not exceeding x is $x + o(\log x)$. Then the a's are the primes.

I have to apologize if my references are sometimes inaccurate but one does not always remember who suggested a problem in a discussion where many things are mentioned in rapid succession. Once the following beautiful problem was attributed to me: Does there exist an infinite sequence of distinct Gaussian primes satisfying $|y_{n+1} - y_n| < C$? I did not remember who told me this but E. Straus cleared this up. The conjecture was told me by Motzkin at the Pasadena number theory meeting 1963 November and it was apparently raised by Basil Gordon and Motzkin. I naturally liked it very much and told it right away to many people, naturally attributing it to Motzkin, but this was later forgotten. Thus the problem is returned to its rightful owners.

The following problem was considered by Graham Sárközi and myself: Let S be a measurable set in the circle $|y| < r$ and assume that no distance between two points of S is an integer. How large can be the measure of S ? Sárközi has the sharpest results, but nothing has been published yet.

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10. Some more unconventional problems

Let $a_1 < a_i < \ldots$ be an infinite sequence of integers. Denote by $f(n)$ the number of solutions of

$$n = \sum_{i=u}^{v} a_i .$$

Is there a sequence for which $f(n) \to \infty$ as $n \to \infty$. This problem can perhaps be rightly criticized as being artificial and in the backwater of Mathematics but it seems very strange and attractive to me. If $a_i = i$ then $f(n)$ is the number of odd divisors of n. I know of no example where $f(n) \geq 2$ for all $n > n_0$. I tried probabilistic methods but they do not seem to work.

Leo Moser and I considered the case where the a_i are primes. We conjectured that $\overline{\lim}\, f(n) = \infty$ and that the density of integers with $f(n) = k$ exists. We do not even know that the upper density of the integers with $f(n) > 0$ is positive.

MacMahon and Andrews (see G. E. Andrews, Amer. Math Monthly 32 (1975), 922-923) consider the following problem. Let $1 = x_1 < x_2 < \ldots$ be the sequence of integers where x_n is the smallest integer which is not the sum of consecutive x_i's (i. e. $x_n \neq \sum_u^v x_i$). Andrews conjectures that

$$x_n = (1+o(1)) \frac{n \log n}{\log\log n} .$$

As far as I know it is not even known whether $x_n/n \to \infty$.

I could not settle this question - all I could do is to ask a few other questions. Let $x_1 < x_2 < \ldots$ be such that no x_n is the sum of consecutive x_i's. Is it true that the density (lower density) of the x_i's is 0? I am not sure about the density but would be very surprised if the lower density would not be 0.

Assume now that all the sums $\sum_u^v x_i$ are distinct. I am now confident that the density of this sequence is 0. It is obvious by a simple averaging process that $x_n > c\, n \log n$ must hold for infinitely many n, thus the lower density is 0. It is not hard to show that for these sequences

$$\sum_{n < x_i < n^2} \frac{1}{x_i} < C$$

holds for an absolute constant C. Perhaps $\Sigma \frac{1}{x_i}$ converges but I do not see how to prove or disprove this.

Let $x_1 = 1 < x_2 < \ldots$ where x_n is the smallest integer for which all the sums $\sum_u^v x_i$ are distinct. A simple counting argument gives $x_n < c n^3$ for all n. I am sure that there is such a sequence with $x_n/n^3 \to 0$. All the problems on finite and infinite B_2 sequences can be asked here too, but almost nothing is known. I am not really sure how difficult these questions are since I did not have time so far to investigate them carefully.

One more question: Let $L(n)$ be the smallest integer so that for every $1 \le a_1 < \ldots < a_k \le n$ there is an m, $n < m \le L(n)$ which is not the sum of consecutive a's. Is it true that $L(n) < C n$?

Recently D. Hofstadter told me several of his problems which were inspired by this question of Ulam. Here is a small sample of his problems: Define a sequence of integers $a_1 < a_2 < \ldots$ as follows: $a_1 = 1$, $a_2 = 2$. If $a_1 < \ldots < a_n$ are already defined, then a_{n+1} is the smallest integer which is the sum of two or more consecutive a's. What is the asymptotic behavior of this sequence?

Another of his problems: Let $a_1 = 2$, $a_2 = 3$. Form all products of two distinct elements of the sequence, subtract 1 and append these elements to the sequence. Repeat this operation indefinitely. Does this sequence have positive density?

A few days ago, Kenneth Rosen told me the following construction: Define a sequence $1 \neq a_1 < a_2 < \ldots$ inductively as follows: $a_1 = 1$, assume that $a_2, \ldots, a_k$ are already defined. Let x be the smallest integer for which the number of solutions of $a_i + a_j \le x$. $1 \le i \le j \le k$ is less than $x - k$ (or $a_0 = 0$, $a_1 = 1$, and the number of solutions of $a_i + a_j \le x$, $0 \le i \le j \le k$, $j > 0$ is less than x). Then put $x = a_{k+1}$. Observe that the number of solutions of $a_i + a_j \le x$ is always $\ge x$. Rosen hoped that the number of solutions of $a_i + a_j \le x$ will then be of the form $x + o(x^{1/4+\varepsilon})$ which would show that my theorem with Fuchs is essentially best possible. Unfortunately, we could not even prove that the number of solutions of $a_i + a_j \le x$ is less than $x + o(x)$. There is little doubt, though, that this and a good deal more is true.

Early in September (of 1976) Harheim considered the following question: Let $a_1, \ldots, a_n$ be a permutation of the integers $1, 2, \ldots, n$. Is it true that there is an n_0 so that for $n > n_0$ the number of distinct sums of the form $\sum_{i=u}^{v} a_i$, $1 \le u \le v \le n$, is less than εn^2? We proved this if $a_i = i$, but could not attack the general case.

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