ON DIFFERENCES AND SUMS OF INTEGERS, II.

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Dedicated to the memory of Christos D. Papakyriakopoulos

1. Throughout this paper, we use the following notations.

$c_1, c_2, \dots$ denote positive absolute constants. We write $e^x = \exp(x)$. $\log_k x$ denotes the k–fold iterated logarithm. If x is a real number, we put $\{x\} = x - [x]$ and we denote the distance of x from the nearest integer (the "norm" of x), by $\|x\|$, i.e., $\|x\| = \min\{x - [x], [x] + 1 - x\}$. For $N = 1,2,\dots$, $\Gamma(N)$ denotes the set of the subsets of $1,2,\dots,N$. The number of the elements of a finite set S is denoted by $|S|$. A, B, ... denote strictly increasing sequences of positive integers. The elements of such sequences are denoted by the corresponding lower case letters, in other words, e.g. $A = \{a_1, a_2, \dots\}$. We write

$$A(n) = \sum_{\substack{a \in A \\ a \leqslant n}} 1 \quad (= |A \cap \{1,2,\dots,n\}|), \quad B(n) = \sum_{\substack{b \in B \\ b \leqslant n}} 1, \dots .$$

If the infinite sequence $B = \{b_1, b_2, \dots,\}$ is such that the equation

$$a_x - a_y = b_z \tag{1}$$

is solvable for any infinite sequence $A = \{a_1, a_2, \dots\}$ of positive lower (asymptotic) density (in other words, B intersects the difference set of each of these sequences A) then we say that B is a *difference intersector set.* (This terminology is due, partly, to R. Tijdeman.) Similarly, if

$$a_x + a_y = b_z \tag{2}$$

is solvable for any infinite sequence A of positive lower density then B is said to be a *sum intersector set.*

We shall use this terminology also for finite sequences $B \subset \Gamma(N)$. In fact, if

$$A(N) > \epsilon N \tag{3}$$

implies the solvability of (1) (if N is large in terms of ϵ) then again, B is said to be a difference intersector set. In the definition of (finite) sum intersector sets, (3) must be replaced by

$$A([N/2]) > \epsilon N.$$

Namely, if $a_u, a_v > [N/2]$ then $a_u + a_v > N$ thus B does not intersect the set of these sums $a_u + a_v$.

In [3] and [5], respectively, the second author showed that both sequences $\{1^2, 2^2, \ldots, z^2, \ldots\}$ and $\{2-1, 3-1, 5-1, \ldots, p-1, \ldots\}$ are difference intersector sets. More exactly, he proved that

Theorem 1 (A. Sárközy, [3]). *If N is large, $A \subset \Gamma(N)$ and*

$$A(N) > c_1 N \frac{(\log_2 N)^{2/3}}{(\log N)^{1/3}} \tag{4}$$

then

$$a_x - a_y = z^2 \quad (z > 0) \tag{5}$$

is solvable.

Theorem 2. (A. Sárközy, [5]). If N is large, $A \subset \Gamma(N)$ and

$$A(N) > c_2 N \frac{(\log_3 N)^3 \log_4 N}{(\log_2 N)^2} \tag{6}$$

then

$$a_x - a_y = p - 1 \tag{7}$$

is solvable.

We guess that Theorem 1 is true even with $N^{1/2+\epsilon}$ on the right hand side of (4) but it seems to be hopeless to prove this at the present time. On the other hand, the second author showed in [4] that the statement of Theorem 1 is not true replacing the right hand side of (4) by

$$N^{1/2} \exp\left\{\left(\frac{1}{2}-\epsilon\right) \frac{\log N \log_3 N}{\log_2 N}\right\} \text{(for any } \epsilon > 0 \text{ and } N > N_0(\epsilon)\text{)}.$$

Also, we guess that the right hand side of (6) in Theorem 2 can be replaced by N^{ϵ} (for any $\epsilon > 0$) or, perhaps, by $(\log N)^{c_3}$. On the other hand, it is easy to see that there exists a constant c_4 such that Theorem 2 is not true replacing the right hand side of (6) by $c_4 \log N$, and the authors conjectured in [2] (see Problem 5) that

$$A(N) / \log N \to +\infty \tag{8}$$

does not imply the solvability of (7), in other words, for $N \to +\infty$, there exist sequences $A \subset \Gamma(N)$ such that (8) holds and (7) is not solvable.

In Part I of this paper (see [1]) the authors proved two general theorems saying that if a sequence $\{b_1, b_2, \ldots, b_i\}$ is well distributed simultaneously among and within all residue classes of small moduli then it must be both difference and sum intersector set. Also, they applied these theorems to investigate the solvability of the equations $\left(\frac{a_x - a_y}{p}\right) = +1$, $\left(\frac{a_u - a_v}{p}\right) = -1$, $\left(\frac{a_r + a_s}{p}\right) = +1$, $\left(\frac{a_t + a_z}{p}\right) = -1$ (where $\left(\frac{a}{p}\right)$ denotes the Legendre symbol) and to show that "almost all" sequences are both difference and sum intersector sets.

The aim of this paper is to continue the investigation of difference and sum intersector sets.

2. In this section, we show that our conjecture concerning the solvability of (7) follows easily from a theorem of A. Schinzel, and we formulate two conjectures related to Theorems 1 and 2, respectively.

Theorem 3. *There exist constants* $c_5\ (>0)$ *and* N_0 *such that if* $N > N_0$ *then there exists a sequence* $A \subset \Gamma(N)$ *for which*

$$A(N) > c_5 \log N \frac{\log_2 N \log_4 N}{(\log_3 N)^2} \tag{9}$$

and (7) is not solvable.

Proof of Theorem 3. Let us denote the smallest prime in the arithmetic progression $kn + \ell$ $(n = 1,2,\ldots)$ by $p(k, \ell)$. In [6], A. Schinzel proved the following theorem:

There exists an absolute constant $c_6 > 0$ such that for every $\ell \neq 0$,

$$p(k,\ell) > c_6\, k \log k \frac{\log_2 k \log_4 k}{(\log_3 k)^2}$$

holds for infinitely many k relatively prime to ℓ.

Applying Schinzel's theorem with $\ell = 1$, we obtain that there exist infinitely many integers k such that

$$p(k,1) > c_6\, k \log k \frac{\log_2 k \log_4 k}{(\log_3 k)^2}. \tag{10}$$

For such a number k, let

$$N = p(k,1) - 1 \tag{11}$$

and

$$A = \left\{ k, 2k, 3k, \ldots, \frac{p(k,1)-1}{k} . k \right\}.$$

Then obviously, $A \subset \Gamma(N)$. Furthermore, $a_x \in A$, $a_y \in A$, $a_x > a_y$ imply that

$$1 \leqslant a_x - a_y < p(k,l) - l, \qquad k / a_x - a_y$$

or in equivalent form,

$$2 \leqslant a_x - a_y + l < p(k,l), \quad a_x - a_y + l \equiv l \pmod{k}$$

By the definition of $p(k,l)$, this implies that

$$a_x - a_y + l \neq p$$

thus in fact, (7) is not solvable.

Finally, we have to show that also (9) holds (for large k). Obviously,

$$\text{(12)} \qquad A(N) = \frac{p(k,l)-l}{k} = \frac{N}{k} \quad .$$

For large k, (10) implies (with respect to (11)) that

$$\text{(13)} \quad N = p(k,l) - l > \frac{c_6}{2} k \log k \frac{\log_2 k \log_4 k}{(\log_3 k)^2} = c_7 k \log k \frac{\log_2 k \log_4 k}{(\log_3 k)^2} \quad .$$

Let us put

$$\text{(14)} \qquad x = c_7 k \log k \frac{\log_2 k \log_4 k}{(\log_3 k)^2} \quad .$$

Then for $k \to +\infty$ we have $\log x \sim \log k$, thus we obtain from (14) that for large k,

$$\left(\frac{1}{c_7} \frac{x}{\log x} \frac{(\log_3 x)^2}{\log_2 x \log_4 x} \sim\right) k < \frac{2}{c_7} \frac{x}{\log x} \frac{(\log_3 x)^2}{\log_2 x \log_4 x} \quad .$$

For large x, this is an increasing function of x, and by (13) and (14), we have $N > x$. Thus

(15) $$k < \frac{2}{c_7} \frac{N}{\log N} \frac{(\log_3 N)^2}{\log_2 N \log_4 N} .$$

(12) and (15) yield that

$$A(N) = \frac{N}{k} > \frac{c_7}{2} \log N \frac{\log_2 N \log_4 N}{(\log_3 N)^2}$$

and this completes the proof of Theorem 3.

As Theorems 1 and 2 show, both sequences $\{1^2, 2^2, \ldots, z^2, \ldots\}$ and $\{2-1, 3-1, \ldots, p-1, \ldots\}$ are difference intersector sets. On the other hand, this sequences are not sum intersector sets. In fact, if $A = \{1, 4, 7, \ldots, 3k+1, \ldots\}$ then

$$\frac{A(N)}{N} \geqslant \frac{1}{3} ,$$

however,

(16) $$a_x + a_y = z^2$$

is not solvable. Similarly, for $A = \{4, 7, \ldots, 3k+1, \ldots\}$, we have

$$\frac{A(N)}{N} \geqslant \frac{1}{3} - \frac{1}{N}$$

but

(17) $$a_x + a_y = p-1$$

is not solvable. We guess that these examples are extremal in the sense that for $\epsilon > 0$, $N > N_0(\epsilon)$,

$$\frac{A(N)}{N} > \frac{1}{3} + \epsilon$$

implies the solvability of both equations (16) and (17).

3. In this section, we prove that if $\alpha(>1)$ is a fixed irrational number then the set

(18) $$B = \{ [\alpha], [2\alpha], \ldots, [n\alpha], \ldots \}$$

is a difference intersector set but it need not be a sum intersector set.

In fact, let β be any real number satisfying $\beta > 1$ and let us put $\alpha = 3\beta$ in (18). Let

$$A = \{ [\beta], [4\beta], \ldots, [(3k-2)\beta], \ldots \}.$$

Then

$$\frac{A(N)}{N} > \frac{1}{3\beta} - \frac{1}{N}.$$

Furthermore, if $a_x = [(3x-2)\beta] \in A$, $a_y = [(3y-2)\beta] \in A$ then we have

$$a_x + a_y \leqslant (3x-2)\beta + (3y-2)\beta = 3(x+y-1)\beta - \beta = (x+y-1)\alpha - \beta < (x+y-1)\alpha - 1 < [(x+y-1)\alpha]$$

and

$$a_x + a_y > (3x-2)\beta - 1 + (3y-2)\beta - 1 = 3(x+y-2)\beta + 2\beta - 2 > (x+y-2)\alpha \geqslant [(x+y-2)\alpha].$$

Thus the sum $a_x + a_y$ lies between the consecutive elements $[(x+y-2)\alpha], [(x+y-1)\alpha]$ of the sequence B, consequently,

$$a_x + a_y = [n\alpha]$$

is not solvable. This proves that in fact, the sequence (18) is not a sum intersector set.

On the other hand, we are going to prove that

Theorem 4. *Let $\alpha > 1$ be any irrational number. Then there exist infinitely many positive integers N such that*

(19) $$A \subset \Gamma(N),$$

(20) $$A(N) > \alpha^{1/2} N^{1/2}$$

imply the solvability of

(21) $$a_x - a_y = [z\alpha].$$

Proof of theorem 4.

The theory of the continued fractions yields that there exist infinitely many positive integers p, q such that

(22) $$0 < \alpha - \frac{p}{q} < \frac{1}{q^2}.$$

We are going to show that the integers N of the form

(23) $$N = pq$$

satisfy the conditions in Theorem 4.

(22) implies that for $1 \leqslant i \leqslant q$,

$$iq\,\alpha > ip$$

and

$$iq\alpha < ip + iq\cdot\frac{1}{q^2} \leqslant ip + 1,$$

hence

(24) $$[iq\,\alpha] = ip \quad \text{for} \quad 1 \leqslant i \leqslant q.$$

For a sequence A sitisfying (19) and (20), let A_j denote the set of those integers a for which $a \in A$ and $a \equiv j \pmod{p}$ hold. Then

(25) $$A = \bigcup_{j=1}^{p} A_j.$$

By (20), (22), (23) and (25), there exists an integer j for which

$$|A_j| \geqslant \frac{A(N)}{p} > \frac{a^{1/2}\ N^{1/2}}{p} > \frac{(\frac{p}{q})^{1/2}\ N^{1/2}}{p} = (\frac{N}{pq})^{1/2} = 1$$

holds, thus $|A_j| \geqslant 2$. Assume that $0 < a_y < a_x, a_x \in A_j$, $a_y \in A_j$. Then obviously,

(26) $$1 \leqslant a_x - a_y \leqslant N = pq$$

and

(27) $$p / a_x - a_y .$$

(24), (26) and (27) yield that (21) holds with $z = iq$ where $i = (a_x - a_y)/p$ and this completes the proof of Theorem 4.

(It can be shown easily that Theorem 4 is near best possible. In fact, the right hand side of (20) can not be replaced by a function f(N) such that $f(N) = o(N^{1/2})$ holds.)

4. In Sections 4, 5 and 6, we shall investigate "sparse" intersector sets. In particular, in this section we discuss the case when the intersector set is finite. We show that for $N \to +\infty$, there exist *difference* intersector sets $B \subset \Gamma(N)$ such that B(N) is bounded; on the other hand, for *sum* intersector sets, $B(N) \to +\infty$ must hold. The first statement is near trivial.

Theorem 5. *If k, d are positive integers, $\epsilon > 0$ is any real number, $N > N_0(k,d,\epsilon)$ and we put $B = \{b_1, b_2, \ldots, b_k\} = \{d, 2d, \ldots, kd\}$, then $A \subset \Gamma(N)$ and*

(28) $$A(N) > (\frac{1}{k+1} + \epsilon) N$$

imply the solvability of

(29) $$a_x - a_y = b_z .$$

Proof of theorem 5.

For $r = 1, 2, \ldots, d$ and $i = 0, 1, 2, \ldots$, we write

$$A_{(r,i)} = A \cap \{ r+i(k+1)d, r+(i(k+1)+1)d, r+(i(k+1)+2)d, \ldots, r+((i+1)(k+1)-1)d \} .$$

Then

$$A \subset \bigcup_{i=0}^{[\frac{N}{(k+1)d}]} \bigcup_{r=1}^{d} A_{(r,i)}. \tag{30}$$

For large N, (28) and (30) imply the existence of r, i such that $|A_{(r,i)}| > 2$. Then there exist a_x, a_y with $a_x \in A_{(r,i)}$, $a_y \in A_{(r,i)}$, $a_x > a_y$. The difference $a_x - a_y$ of these numbers can be written in the form jd where $1 \leqslant j \leqslant k$, thus $a_x - a_y \in B$ which proves the solvability of (29).

Theorem 6. *Let*

$$0 < \epsilon < \frac{1}{4} \tag{31}$$

If $N > N_0(\epsilon)$, $B \subset \Gamma(N)$ *and*

$$B(N) < \frac{1}{2 \log 1/\epsilon} \log N \tag{32}$$

then there exists a sequence $A \subset \Gamma([N/2])$ *such that*

$$A([N/2]) > (\frac{1}{2} - \epsilon)\ [\frac{N}{2}] \tag{33}$$

holds and

$$a_x + a_y = b_z \tag{34}$$

is not solvable.

Proof of Theorem 6.

We shall need the following well–known theorem on simultaneous approximation:

Let k be a positive integer, $\alpha_1, \alpha_2, ..., \alpha_k$, Q real numbers, $Q \geqslant 1$. Then there exist integers q, $p_1, p_2, ..., p_k$ such that

$$1 \leqslant q \leqslant Q \tag{35}$$

and

$$|\alpha_i - \frac{p_i}{q}| \leqslant \frac{1}{qQ^{1/k}}.$$

Let us apply this theorem with $k = B(N)$, $\alpha_i = \frac{b_i}{[N/2]}$ ($b_1, b_2, \ldots, b_{B(N)}$ denote the elements of a set B satisfying (32)) and $Q = \frac{\epsilon}{4}[N/2]$ (for large N, $Q \geqslant 1$ holds trivially). We obtain that there exist integers $q, p_1, p_2, \ldots, p_k$ such that (35) holds and

$$\left| \frac{b_i}{[N/2]} - \frac{p_i}{q} \right| < \frac{1}{qQ^{1/k}},$$

whence

$$\left| \frac{q}{[N/2]} b_i - p_i \right| < \frac{1}{Q^{1/k}},$$

(36)
$$\left\| \frac{q}{[N/2]} b_i \right\| < \frac{1}{Q^{1/k}}.$$

Let us define the positive integers r, s by

(37)
$$\frac{r}{s} = \frac{q}{[N/2]}, \quad (r,s) = 1.$$

Then with respect to (35),

(38)
$$\frac{1}{s} \leqslant \frac{r}{s} = \frac{q}{[N/2]} \leqslant \frac{Q}{[N/2]} = \frac{\epsilon}{4}.$$

For $i = 0, 1, \ldots, s-1$, let A_i denote the set of those integers a for which $1 \leqslant a \leqslant [N/2]$ and

$$\left\{ \frac{q}{[N/2]} a \right\} = \left\{ \frac{r}{s} a \right\} = \frac{i}{s}$$

hold. Then obviously,

$$\{1, 2, \ldots, [N/2]\} = \bigcup_{i=0}^{s-1} A_i$$

and

$$|A_0| = |A_1| = \dots = |A_{s-1}|$$

(note that $s/[N/2]$ by (37)), hence

$$A_i = \frac{[N/2]}{s} \qquad \text{(for } i = 0, 1, \dots, s-1). \tag{39}$$

Let us define the sequence A by

$$A = \bigcup_{\frac{1}{2Q^{1/k}} < \frac{i}{s} < \frac{1}{2} - \frac{1}{2Q^{1/k}}} A_i . \tag{40}$$

Then $A \subset \Gamma([N/2])$ holds trivially. We are going to show that also (33) holds and (34) is not solvable.

In fact, $a_x \in A$, $a_y \in A$ imply that

$$\frac{1}{2Q^{1/k}} < \left\{ \frac{q}{[N/2]} a_x \right\} < \frac{1}{2} - \frac{1}{2Q^{1/k}}$$

and

$$\frac{1}{2Q^{1/k}} < \left\{ \frac{q}{[N/2]} a_y \right\} < \frac{1}{2} - \frac{1}{2Q^{1/k}},$$

thus

$$\left\{ \frac{q}{[N/2]} (a_x + a_y) \right\} = \left\{ \frac{q}{[N/2]} a_x \right\} + \left\{ \frac{q}{[N/2]} a_y \right\} > \frac{1}{2Q^{1/k}} + \frac{1}{2Q^{1/k}} =$$

$$= \frac{1}{Q^{1/k}}$$

and

$$\left\{ \frac{q}{[N/2]} (a_x + a_y) \right\} = \left\{ \frac{q}{[N/2]} a_x \right\} + \left\{ \frac{q}{[N/2]} a_y \right\} < \left(\frac{1}{2} - \frac{1}{2Q^{1/k}} \right) +$$

$$+ \left(\frac{1}{2} - \frac{1}{2Q^{1/k}} \right) = 1 - \frac{1}{Q^{1/k}}$$

$$\| \frac{q}{[N/2]} (a_x + a_y) \| > \frac{1}{Q^{1/k}}.$$

In view of (36), this inequality yields that none of the integers $b_z \in B$ can be written in the form $a_x + a_y$, in other words, (34) is not solvable.

Finally, by (38), (39) and (40), we have

$$A([N/2]) = |A| = \sum_{\frac{1}{2Q^{1/k}} < \frac{i}{s} < \frac{1}{2} - \frac{1}{2Q^{1/k}}} |A_i| = \tag{41}$$

$$= \frac{[N/2]}{s} \Big(\sum_{0 \leqslant i \leqslant \frac{1}{2}s} 1 - \sum_{0 \leqslant i \leqslant \frac{1}{2Q^{1/k}}} 1 - \sum_{(\frac{1}{2} - \frac{1}{2Q^{1/k}})s \leqslant i \leqslant \frac{1}{2}s} 1 \Big)$$

$$> \frac{[N/2]}{s} \Big(\frac{1}{2}s - \Big(\frac{1}{2Q^{1/k}} s + 1 \Big) - \Big(\frac{1}{2Q^{1/k}} s + 1 \Big) \Big) = [N/2] \Big(\frac{1}{2} - \Big(\frac{1}{Q^{1/k}} + \frac{2}{s} \Big) \Big)$$

$$\geqslant [N/2] \Big(\frac{1}{2} - \Big(\frac{1}{Q^{1/k}} + \frac{2}{s} \Big) \Big) \geqslant [N/2] \Big(\frac{1}{2} - \frac{1}{Q^{1/k}} - \frac{\epsilon}{2} \Big).$$

With respect to (32), for fixed ϵ and $N \to +\infty$ we have

$$\frac{1}{Q^{1/k}} = \exp\Big(-\frac{1}{k} \log Q\Big) = \exp\Big(-\frac{1}{B(N)} \log \frac{\epsilon}{4} [N/2] \Big)$$

$$< \exp\Big(-\frac{2 \log 1/\epsilon}{\log N} \log \frac{\epsilon}{4} [N/2]\Big) = \exp\Big(\frac{2 \log \epsilon}{\log N} (\log N + (O(1))\Big) =$$

$$= \exp(2 \log \epsilon + o(1)) = (1+o(1))\epsilon^2.$$

Thus in view of (31), we have

$$\frac{1}{Q^{1/k}} < 2\epsilon^2 \leqslant \frac{\epsilon}{2} \tag{42}$$

for large enough N. (41) and (42) yield (33) and the proof of Theorem 6 is completed.

5. In this section, we investigate *infinite* (but sparse) intersector sets.

R. Tijdeman raised the conjecture (in a letter written to the first author) that if $B = \{b_1, b_2, \ldots\}$ is an infinite difference intersector set then

(43) $$\lim_{k\to\infty} \inf \frac{b_{k+1}}{b_k} = 1$$

must hold. First we prove this conjecture*.

Theorem 7. *If $\Delta > 1$, $B = \{b_1, b_2, \ldots, b_k, \ldots\}$ is a strictly increasing infinite sequence of positive integers and*

(44) $$\inf_{k=1,2,\ldots} \frac{b_{k+1}}{b_k} \geqslant \Delta \quad (>1),$$

then there exists a strictly increasing sequence $A = \{a_1, a_2, \ldots, a_k, \ldots\}$ of positive integers such that

(45) $$\lim_{n\to\infty} \inf \frac{A(N)}{N} \geqslant \exp\left(-\left(\frac{\log 3}{\log \Delta} + 1\right)\log 24\right)$$

and the equations

(46) $$a_x - a_y = b_z,$$

(47) $$a_u + a_v = b_t$$

are not solvable.

We shall need two lemmas.

Lemma 1. *If*

(48) $$0 < \gamma \leqslant 1,$$

and $d_1, d_2, \ldots, d_k$ are positive integers for which

(49) $$\frac{d_{k+1}}{d_k} \geqslant 2 + \gamma \qquad \text{for } k = 1, 2, \ldots$$

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* Note added in proof. In the meantime, this conjecture has been proved independently also by C.L. Stewart and R. Tijdeman. (Unpublishect yet).

then there exists a real number a such that

$$\|d_k \alpha\| \geq \frac{\gamma}{6} \quad \text{for} \quad k = 1,2, \dots . \tag{50}$$

Proof of Lemma 1.

We are going to construct intervals $I_1, I_2, \dots, I_k, \dots$ satisfying the following conditions:

(i) The intervals $I_1, I_2, \dots, I_k, \dots$ are closed.

(ii) $I_{k+1} \subset I_k$ for $k = 1,2, \dots$.

(iii) $x \in I_k$ implies that

$$\| d_k x \| \geq \frac{\gamma}{6}$$

(for $k = 1,2,\dots$).

(iv) The of the interval $I_k = [u_k, v_k]$ is

$$v_k - u_k = \left(1 - \frac{\gamma}{3}\right) \frac{1}{d_k} .$$

Obviously, the interval $I_1 = \left[\frac{\gamma}{6} \frac{1}{d_1}, \left(1 - \frac{\gamma}{6}\right) \frac{1}{d_1} \right]$ satisfies (i), (iii) and (iv). Let us assume now that the intervals $I_1, I_2, \dots, I_k = [u_k, v_k]$ have been defined. Then I_{k+1} can be defined in the following way:

With respect to (48), (49) and (iv), we have

$$d_{k+1} v_k - d_{k+1} u_k = (v_k - u_k) d_{k+1} = \left(1 - \frac{\gamma}{3}\right) \frac{d_{k+1}}{d_k} \geq \left(1 - \frac{\gamma}{3}\right)(2+\gamma) =$$

$$= 2 + \frac{\gamma}{3}(1 - \gamma) \geq 2.$$

Thus there exists an integer t such that

$$d_{k+1} u_k \leq t < t+1 \leq d_{k+1} v_k$$

or in equivalent form,

$$u_k \leqslant \frac{t}{d_{k+1}} < \frac{t+1}{d_{k+1}} \leqslant v_k. \tag{51}$$

Let

$$I_{k+1} = [u_{k+1}, v_{k+1}] = \left[\frac{t+\frac{\gamma}{6}}{d_{k+1}}, \frac{t+1-\frac{\gamma}{6}}{d_{k+1}} \right].$$

Then the interval I_{k+1} is closed. (ii) follows from (48) and (51). Furthermore, for $x \in I_{k+1}$, we have

$$d_{k+1}x \geqslant d_{k+1} \frac{t+\frac{\gamma}{6}}{d_{k+1}} = t+\frac{\gamma}{6}$$

and

$$d_{k+1}x \leqslant d_{k+1} \frac{t+1-\frac{\gamma}{6}}{d_{k+1}} = t+1-\frac{\gamma}{6}$$

which implies that (iii) holds (with k+1 in place of k). Finally,

$$v_{k+1}-u_{k+1} = \frac{t+1-\frac{\gamma}{6}}{d_{k+1}} - \frac{t+\frac{\gamma}{6}}{d_{k+1}} = \frac{1-\frac{\gamma}{3}}{d_{k+1}}$$

thus also (iv) holds (with k+1 in place of k) and this completes the proof of the existence of intervals $I_1, I_2, \ldots, I_k, \ldots$ satisfying (i)–(iv).

By (i) and (ii), the intersection of the intervals $I_1, I_2, \ldots, I_k, \ldots$ can not be empty. Thus there exists a real number α such that $\alpha \in I_k$ for $k = 1,2,\ldots$. By (iii), this number α satisfies (50) for all k and the proof of Lemma 1 is completed.

Lemma 2. *If*

$$0 < \delta < 1, \tag{52}$$

k is a positive integer, $a_1, a_2, \ldots, a_k$ *are any real numbers, then for* $N > N_0(\delta, k, a_1, \ldots, a_k)$, *the number of the integers n satisfying*

$$1 \leqslant n \leqslant N \tag{53}$$

and

$$\|n\alpha_i\| < \delta \tag{54}$$

(simultaneously for $i = 1, 2, \ldots, k$) is greater than $(\frac{\delta}{2})^k N$.

Proof of Lemma 2.

Let us define the positive integer t by

$$t > \frac{1}{\delta} \geqslant t-1. \tag{55}$$

For $1 \leqslant n \leqslant N$, let P_n denote the point $(\{n\alpha_1\}, \{n\alpha_2\}, \ldots, \{n\alpha_k\})$ in the k–dimensional Euclidean space. Let us form all the k–tuples $(r_1, r_2, \ldots, r_k)$ such that r_1, $r_2, \ldots, r_k$ are integers and $0 \leqslant r_i \leqslant t-1$ for $i = 1, 2, \ldots, k$. For each of these k–tuples $(r_1, r_2, \ldots, r_k)$, we form the k–dimensional cube consisting of those points $(x_1, x_2, \ldots, x_k)$ which satisfy $\frac{r_i}{t} \leqslant x < \frac{r_i+1}{t}$ for $i = 1, 2, \ldots, k$. The number of these number of these cubes is t^k, and each of the N points $P_1, P_2, \ldots, P_N$ is contained in one of these cubes. Thus one of these cubes contains at least $\frac{N}{t^k}$ of the points $P_1, P_2, \ldots, P_N$. In other words, there exist integers $n_1, n_2, \ldots, n_s$ such that P_{n_1}, P_{n_2}, $\ldots, P_{n_s}$ lie in the same cube and

$$s \geqslant \frac{N}{t^k} \quad . \tag{56}$$

We may assume that $n_1 < n_2 < \cdots < n_s$. Then

$$\|(n_j - n_1)\alpha_i\| = \|n_j\alpha_i - n_1\alpha_i\| = \|\{n_j\alpha_i\} - \{n_1\alpha_i\}\| \leqslant |\{n_j\alpha_i\} - \{n_1\alpha_i\}| <$$
$$< \frac{1}{t} < \delta$$

simultaneously for $i = 1, 2, \ldots, k$ and for all $2 \leqslant j \leqslant s$, and obviously,

$0 < n_j - n_1 < n_j \leqslant N$ for $2 \leqslant j \leqslant s$. Thus the integers $n = n_2 - n_1$, $n_3 - n_1, \ldots$, $n_s - n_1$ satisfy (53) and (54), and in view of (52), (55) and (56), their number is at least

$$s-1 \geqslant \frac{N}{t^k} - 1 \geqslant \frac{N}{(\frac{1}{\delta}+1)^k} - 1 > \frac{N}{(\frac{2}{\delta})^k} = (\frac{\delta}{2})^k N$$

for large enough N which completes the proof of Lemma 2.

Completion of the proof of Theorem 7.

Assume that the sequence B satisfies (44) and define the positive integer k by

$$\Delta^k \geqslant 3 > \Delta^{k-1} . \tag{57}$$

Then

$$k < \frac{\log 3}{\log \Delta} + 1. \tag{58}$$

For $i = 1,2,..., k$, let $B_i = \{ b_i, b_{i+k}, b_{i+2k}, ..., b_{i+jk}, ... \}$. Then obviously,

$$B = \bigcup_{i=1}^{k} B_i \tag{59}$$

and with respect to (44) and (57),

$$\frac{b_{i+(j+1)k}}{b_{i+jk}} = \prod_{n=i+jk}^{i+(j+1)k-1} \frac{b_{n+1}}{b_n} \geqslant \prod_{n=i+jk}^{i+(j+1)k-1} \Delta = \Delta^k \geqslant 3.$$

Thus for each of the sequences $B_1, B_2, ..., B_k$, we may apply Lemma 1 putting $\gamma = 1$ and with the sequence B_i in place of $d_1, d_2, ...$. We obtain that there exist real numbers $\alpha_1, \alpha_2, ..., \alpha_k$ such that

$$\| b\, \alpha_i \| \geqslant \frac{1}{6} \quad \text{for } b \in B_i \tag{60}$$

(where $i = 1, 2, ..., k$).

Let A denote the set of those positive integers a for which

$$\| a\, \alpha_i \| < \frac{1}{12} \tag{61}$$

holds simultaneously for $i = 1, 2, \ldots, k$. Applying Lemma 2 with $\delta = \frac{1}{12}$, we obtain with respect to (58) that

$$A(N) > \left(\frac{1}{24}\right)^k N > N \exp\left(-\left(\frac{\log 3}{\log \Delta} + 1\right)\log 24\right)$$

which proves (45).

To complete the proof, we have to show that (46) and (47) are not solvable. By (61), $a_x \in A$, $a_y \in A$, $a_u \in A$, $a_v \in A$ imply that

$$\|(a_x - a_y)\alpha_i\| \leqslant \|a_x \alpha_i\| + \|a_y \alpha_i\| < \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

and

$$\|(a_u + a_v)\alpha_i\| \leqslant \|a_u \alpha_i\| + \|a_v \alpha_i\| < \frac{1}{12} + \frac{1}{12} = \frac{1}{6}.$$

By (60), this implies that $a_x - a_y \notin B_i$ and $a_u + a_v \notin B_i$ for $i = 1, 2, \ldots, k$, thus by (59), $a_x - a_y \notin B$ and $a_u + a_v \notin B$ which completes the proof of our theorem.

6. It can be shown easily that for *difference* intersector sers, Theorem 7 is best possible. In fact, let B denote a sequence of the form

$$B = \{b_1, b_2, \ldots\} = \bigcup_{i=1}^{+\infty} \{n_i, n_i + 1, \ldots, n_i + j_i\}$$

where $n_i \to +\infty$ rapidly and $j_i \to +\infty$ slowly. As Theorem 5 shows, this set B is a difference intersector set, and obviously,

(62) $$\frac{b_{k+1}}{b_k} > 1 + \epsilon_k$$

where $\epsilon_k \to 0$ arbitrary slowly.

On the other hand, we do not know whether Theorem 6, also Theorem 7 for sum intersector sets are best possible. In other words, the following problems can be raised:

(i) Is it true that if $\lim_{N \to +\infty} f(N) = +\infty$ then for $\epsilon > 0$, $N > N_0(\epsilon)$, there exists a sequence $B \subset \Gamma(N)$ such that

$$B(N) < f(N) \log N$$

holds, and $A \subset \Gamma([N/2])$,

$$A([N/2]) > \epsilon [N/2]$$

imply the solvability of (34)?

(ii) Is it true that if $\epsilon_k \to 0$ (and $\epsilon_k > 0$) then there exists an infinite sequence $B = \{b_1, b_2, \ldots, b_k, \ldots\}$ such that (62) holds, and

$$\liminf_{N \to +\infty} \frac{A(N)}{N} > 0$$

implies the solvability of (47)?

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