NONBASES OF DENSITY ZERO NOT CONTAINED IN MAXIMAL NONBASES

PAUL ERDŐS AND MELVYN B. NATHANSON

Abstract

A sequence $A = \{a_i\}$ of non-negative integers is a basis if every sufficiently large integer n can be written in the form $n = a_i + a_j$ with $a_i, a_j \in A$. If A is not a basis, then A is called a nonbasis. The nonbasis A is maximal if $A \cup \{b\}$ is a basis for every $b \notin A$. We construct a nonbasis A of density zero, in particular, with $A(x) = O(\sqrt{x})$, such that A cannot be imbedded as a subset of any maximal nonbasis.

A set A of non-negative integers is a *basis* if every sufficiently large integer n can be written in the form $n = a_i + a_j$ with $a_i, a_j \in A$. If infinitely many integers n cannot be represented in the form $n = a_i + a_j$, then A is a *nonbasis*. The set A is a *maximal nonbasis* if A is a nonbasis, but $A \cup \{b\}$ is a basis for every non-negative integer $b \notin A$.

Nathanson [3] asked if every nonbasis is a subset of a maximal nonbasis. Recently, Hennefeld [2] observed that the set A consisting of $\{1\}$ together with all non-negative even integers except those of the form 2^k with $k \geqslant 1$ is a nonbasis that is not a subset of any maximal nonbasis. This set A has density 1/2. On the other hand, Erdős and Nathanson [1] proved that if A is a nonbasis such that $A \cup F$ is a nonbasis for every finite set F of non-negative integers, then A is a subset of some maximal nonbasis. In particular, if A has density 0 and $2A = \{a_i + a_j \mid a_i, a_j \in A\}$ has density strictly less then 1, then A is a subset of a maximal nonbasis. The question remains whether every nonbasis of density 0 is a subset of a maximal nonbasis. If A is a set of non-negative integers, let $A(x)$ denote the number of elements of A not exceeding x. In this note we prove the following best possible result: There exists a nonbasis A with

$$A(x) = O(\sqrt{x})$$

which is not a subset of any maximal nonbasis.

LEMMA. *Let $\{Q_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of odd positive integers $Q_k = 2q_k + 1$ such that*

$$Q_k > 2\left(\sum_{j=1}^{k-1} Q_j\right)^2.$$

Let $A' \subset [0, q_1] \cup \bigcup_{k=2}^{\infty} [Q_{k-1}+1, q_k]$ be a set of integers such that $2A' = \mathbb{N} \setminus \{Q_k\}_{k=1}^{\infty}$ and

$$A' \cap [Q_{k-1}+1, q_k] \subsetneqq [Q_{k-1}+1, q_k]$$

for all sufficiently large k. Then there exists a nonbasis A with $A' \subset A$ such that $A(x) = A'(x) + O(\sqrt{x})$ and A is not a subset of a maximal nonbasis.

Proof. Let $Q_0 = -1$, and let $A_k' = A' \cap [Q_{k-1}+1, q_k]$ for all $k \geqslant 1$. Then $A' = \bigcup_{k=1}^{\infty} A_k'$, and $A_k' \subsetneqq [Q_{k-1}+1, q_k]$ for all $k > k_0 \geqslant 1$.

Received 29 July, 1976.

We shall construct sets A_k such that $A_k' \subset A_k \subset [Q_{k-1}+1, Q_k]$. Let $A_k = A_k'$ for $k \leqslant k_0$. Suppose A_j has been determined for $j < k$. Choose $b_k \in [Q_{k-1}+1, q_k] \setminus A_k'$. Define $M_k \subset [q_k+1, Q_k]$ by

$$M_k = \{Q_k - b_k\} \cup \{Q_k - x \mid x \in [0, Q_{k-1}] \text{ and } x \notin \{b_j\}_{j<k} \cup \bigcup_{j<k} A_j\}.$$

Let $A_k = A_k' \cup M_k$, and let $A = \bigcup_{k=1}^{\infty} A_k$.

Clearly, if $Q_k - x \in [q_k+1, Q_k] \cap A$, then $x \notin A$, and so $Q_k \notin 2A$. Therefore, $2A = 2A' = \mathbb{N} \setminus \{Q_k\}_{k=1}^{\infty}$, and A is a nonbasis.

We shall determine all sets W such that $A \cup W$ is a nonbasis. Let $B = \{b_k\}_{k>k_0}$. Then $A \cap B = \varnothing$. If $c \notin A \cup B$, then $Q_k - c \in M_k \subset A$ for all sufficiently large k, and so $Q_k \in 2(A \cup \{c\})$. Therefore, if $A \cap W = \varnothing$ and $A \cup W$ is a nonbasis, then $W \subset B$, and so $W = B_I = \{b_k\}_{k \in I}$, where I is a subset of $\{k\}_{k>k_0}$.

If $Q_k \in 2(A \cup B)$, then $k > k_0$ and $Q_k = x + y$ for some $x, y \in A \cup B$ with $x < y$. Then $y \in [q_k+1, Q_k]$. But $B \cap [q_k+1, Q_k] = \varnothing$ and

$$A \cap [q_k+1, Q_k] = M_k = \{Q_k - b_k\} \cup \{Q_k - x \mid x \in [0, Q_{k-1}] \setminus (A \cup B)\}.$$

Since $x \in A \cup B$, it follows that $y = Q_k - b_k$ and $x = b_k$. Therefore, if $B_I \subset B$, then $Q_k \in 2(A \cup B_I)$ if and only if $k \in I$. Therefore, $A \cup B_I$ is a nonbasis if and only if I is a subset of $\{k\}_{k>k_0}$ whose complement in $\{k\}_{k>k_0}$ is infinite. Since there is no such maximal set I, there is no maximal subset B_I of B such that $A \cup B_I$ is a maximal nonbasis. Therefore, A is not contained in a maximal nonbasis.

Finally, we compute $A(x) - A'(x)$. Clearly, $|M_k| \leqslant Q_{k-1}$ and $|A_k| \leqslant |A_k'| + Q_{k-1}$ for all k. Let $Q_{k-1} < x \leqslant Q_k/2$. Then

$$\begin{aligned} A(x) &\leqslant A'(x) + \sum_{j=k_0+1}^{k-1} |M_j| \\ &\leqslant A'(x) + \sum_{j=1}^{k-2} Q_j \\ &< A'(x) + \sqrt{Q_{k-1}} \\ &< A'(x) + \sqrt{x}. \end{aligned}$$

Let $Q_k/2 < x \leqslant Q_k$. Then

$$\begin{aligned} A(x) &\leqslant A'(x) + \sum_{j=k_0+1}^{k} |M_j| \\ &\leqslant A'(x) + \sum_{j=1}^{k-1} Q_j \\ &< A'(x) + \sqrt{(Q_k/2)} \\ &< A'(x) + \sqrt{x}. \end{aligned}$$

Therefore, $A(x) = A'(x) + O(\sqrt{x})$.

THEOREM. *There exists a nonbasis A with $A(x) = O(\sqrt{x})$ which is not a subset of a maximal nonbasis.*

Proof. In [4], Nathanson constructed a set A' satisfying the conditions of the Lemma, and also $A'(x) = O(\sqrt{x})$. Applying the Lemma to this set A', we obtain a nonbasis A that is not contained in a maximal nonbasis and that satisfies

$$A(x) = A'(x) + O(\sqrt{x}) = O(\sqrt{x}).$$

Remark. Hennefeld [2] has proved that the set consisting of $\{1\}$ together with all non-negative multiples of h except the powers h^n with $n \geqslant 1$ is a nonbasis of order h which is not a subset of a maximal nonbasis of order h. It would be of interest to construct a nonbasis A of order h with $A(x) = O(x^{1/h})$ such that A is not a subset of a maximal nonbasis of order h.

References

1. P. Erdős and M. B. Nathanson, " Maximal asymptotic nonbases ", *Proc. Amer. Math. Soc.*, 48 (1975), 57–60.
2. J. Hennefeld, " Asymptotic nonbases which are not subsets of maximal asymptotic nonbases ", preprint.
3. M. B. Nathanson, " Minimal bases and maximal nonbases in additive number theory ", *J. Number Theory*, 6 (1974), 324–333.
4. M. B. Nathanson, " s-maximal nonbases of density zero ", *J. London Math. Soc.*, (2), 15 (1977), 29–34.

Mathematical Institute,
Hungarian Academy of Sciences,
Budapest,
Hungary.

Department of Mathematics,
Southern Illinois University,
Carbondale, Illinois 62901,
U.S.A.