JOURNAL OF COMBINATORIAL THEORY, Series B 23, 139–142 (1977)

Hamiltonian Cycles in Regular Graphs of Moderate Degree

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Received October 25, 1976

In this paper we prove that if k is an integer no less than 3, and if G is a two-connected graph with $2n - a$ vertices, $a \in \{0, 1\}$, which is regular of degree $n - k$, then G is Hamiltonian if $a = 0$ and $n \geqslant k^2 + k + 1$ or if $a = 1$ and $n \geqslant 2k^2 - 3k + 3$.

We use the notation and terminology of [1]. Gordon [4] has proved that there are only a small number of exceptional graphs with $2n$ vertices which are not Hamiltonian when all vertices have degree $n - 1$ or more. The present authors proved [3] that if G is a two-connected graph with $2n$ vertices which is regular of degree $n - 2$ and if $n \geqslant 6$, then G is Hamiltonian. We now partially extend that result to regular graphs of degree $n - k$, $k \geqslant 3$.

Throughout this paper we suppose that G is a graph with $2n - a$ vertices, with $a \in \{0, 1\}$, which is two connected and regular of degree $n - k$, where k is an integer no less than three. Let P be a longest cycle in G, choose a direction around P, let $R = V(G) - V(P)$, and let $r = |R|$. For the lemmas, suppose $r \geqslant 1$. By a theorem of Dirac [2], $l(P) \geqslant 2n - 2k$. For $v \in R$, let C_v be the set of vertices of P adjacent to v, let A_v be the set of vertices of P immediately preceding elements of C_v in the ordering of P, and let B_v be the set of vertices of P immediately following elements of C_v. The first lemma is trivial.

LEMMA 1. *Let v and w be in R. Then v is not adjacent to any vertex in $A_v \cup B_v$, A_v and B_v are independent sets of vertices, and w is joined to at most one vertex of A_v and to at most one vertex in B_v.*

LEMMA 2. *If $n \geqslant 3k + 2 - a$, then R is independent.*

Proof. Let Q be a longest path in a component of R and suppose $\ell(Q) \geqslant 1$. Let v and w be the ends of Q and let $d = \max\{\deg_{\langle R\rangle} v, \deg_{\langle R\rangle} w\}$. Then $\ell(Q) \geqslant d$. Thus Q contains at least $d + 1$ vertices. Going around P, let there be t occurrences of a vertex y joined to one of v or w and followed (not necessarily immediately) by a vertex z joined to the other of v and w; then there are at least $d + 1$ vertices between y and z on P which are joined to neither v nor w, for otherwise P could be extended. Thus $2n - r - a = \ell(P) \geqslant$ number of edges from v to P + number of edges from w to P + number of vertices of P joined to v and/or $w + t(d - 1) \geqslant 3n - 3k - 3d + td - t$. Since v and w are both joined to vertices of P, $t \geqslant 2$. Further, $1 \leqslant d \leqslant r - 1$. Thus $1 - d \leqslant 0$. It follows that $n \leqslant 3k + 1 - a$. But $n \geqslant 3k + 2 - a$, so $\ell(Q) = 0$ and R is independent.

Now we fix v and let $A = A_v$, $B = B_v$, and $C = C_v$. Let $X = V(P) - (A \cup B \cup C)$ and let $s = |A - B| = |B - A|$. It is easy to see that $s \geqslant 1$ when $k \geqslant 3$. By Lemma 2, $|A| = |B| = |C| = n - k$ and $|X| = 2k - (r + s) - a$. Since $|X| \geqslant 0$, $r + s \leqslant 2k - a$.

LEMMA 3. *If $n \geqslant 3k + 2 - a$, then $r \leqslant k - a$.*

Proof. Let d be the number of edges from R to B. Then $d \leqslant r - 1$ by Lemma 1. Also by Lemma 1, B is independent. Thus there are $(n - k)(n - k + r) - 2d$ edges from $R \cup B$ to the other $n + k - r - a$ vertices of G. Since G has $(2n - a)(n - k)/2$ edges, $(n - k)(n - k + r) - 2d + d \leqslant (2n - a)(n - k)/2$, from which we get $r \leqslant k - \frac{1}{2}a + (k - \frac{1}{2}a - 1)/(n - k - 1)$. Since r is an integer and $n \geqslant 3k + 2 - a$, $r \leqslant k - a$.

LEMMA 4. *If $n \geqslant k^2 + k + 1$, then $r + s \leqslant k$.*

Proof. Suppose $r + s > k$. By Lemmas 1 and 2, $|E(\langle A \cup B \cup R\rangle)| \leqslant s^2 + 2(r - 1)$. Since $|A \cup B \cup R| = n - k + r + s$, there are at least $(n - k + r + s)(n - k) - 2(s^2 + 2r - 2)$ edges from $A \cup B \cup R$ to $C \cup X$; further, $|C \cup X| = n + k - r - s - a$. Thus

$$(n - k + r + s)(n - k) - 2(s^2 + 2r - 2) \leqslant (n + k - r - s - a)(n - k),$$

whence (using the assumption that $r + s \geqslant k + 1$),

$$n \leqslant k + [(s^2 + 2r - 2)/(r + s - k + \tfrac{1}{2}a)].$$

Denoting this upper bound for n by $f(a, k, r, s)$, holding a, k, and r constant, and recalling that $k + 1 - r \leqslant s \leqslant 2k - a - r$, we find that $f(a, k, r, k + 1 - r)$ is a maximum for f except when $a = 1$ and the pair (k, r) is in $\{(3, 1), (3, 2), (4, 2), (4, 3), (5, 3), (5, 4)\}$. But in these exceptional cases, $f(1, k, r, s) \leqslant k^2 + k$. Further, in all other cases as r ranges through $[1, k]$, treating the cases $a = 0$ and $a = 1$ separately and holding k constant, we get $f(a, k, r, s) \leqslant k^2 + k$. The lemma follows.

LEMMA 5. *Let X_0 be the subset of X such that the elements of X_0 are adjacent to no vertices of $A \cap B$. Then*

(1) *if $a = 0$, $|X_0| \geqslant k - r - s + 1$; and*
(2) *if $a = 1$, $|X_0| \geqslant k - r - s$.*

Proof. There are s intervals on P in which vertices of X might be found. Number these intervals as $1, 2, ..., s$ with m_i elements of X in interval i in such a way that $m_1, m_2, m_3, ..., m_e$ are even and $m_{e+1}, m_{e+2}, ..., m_s$ are odd, with $e \geqslant 0$. It is easily seen that if two vertices of X which are successive around P are both joined to elements of $A \cap B$, then there is a cycle of G larger than P. Hence at least the smallest number of nonconsecutive elements of the sequence of vertices in X in interval i, or $\{(m_i - 1)/2\}$, are not joined to any vertex in $A \cap B$. Thus

$$|X_0| \geqslant \sum_{i=1}^{e} \left(\frac{m_i - 1}{2} + \frac{1}{2}\right) + \sum_{i=e+1}^{s} \frac{m_i - 1}{2}$$
$$= \frac{1}{2}|X| - \frac{1}{2}(s - e) \geqslant \frac{1}{2}(2k - r - 2s - a).$$

If $a = 0$, $|X_0| \geqslant k - r - s + \frac{1}{2}r \geqslant k - r - s + \frac{1}{2}$ since $r \geqslant 1$. But $|X_0|$, k, r, and s are integers, so $|X_0| \geqslant k - r - s + 1$. If $a = 1$, $|X_0| \geqslant k - r - s + \frac{1}{2}r - 1 \geqslant k - r - s - \frac{1}{2}$, whence $|X_0| \geqslant k - r - s$.

THEOREM. *Suppose $k \geqslant 3$. Then G is Hamiltonian if*

(a) *$a = 0$ and $n \geqslant k^2 + k + 1$, or*
(b) *$a = 1$ and $n \geqslant 2k^2 - 3k + 3$.*

Proof. Suppose G is not Hamiltonian. By Lemma 4, $r + s \leqslant k$. By Lemma 5 and the definitions, $|A \cup B \cup R \cup X_0| \geqslant n + 1 - a$. Choose a subset X_0' of X_0 such that $|A \cup B \cup R \cup X_0'| = n + 1 - a$. By the definitions and Lemmas 1 and 2, we have at most

	edges from	to
s^2	A	B
$r - 1$	A	R
$s(k - r - s + 1 - a)$	A	X_0'
$r - 1$	B	R
$s(k - r - s + 1 - a)$	B	X_0'
$(r - 1)(k - r - s + 1 - a)$	R	X_0'
$\frac{1}{2}(k - r - s - a)(k - r - s + 1 - a)$	X_0'	X_0'

in G and no other edges in $\langle A \cup B \cup R \cup X_0' \rangle$. Thus there are at least

$$(n+1-a)(n-k) - 2\{s^2 + 2r - 2 + 2s(k-r-s+1-a) + (r-1)(k-r-s+1-a) + \tfrac{1}{2}(k-r-s-a)(k-r-s+1-a)\}$$

edges from $A \cup B \cup R \cup X_0'$ to $(C \cup X) - X_0'$. Since this number is less than or equal to $(n-1)(n-k)$, we get

$$n \leqslant k + \frac{2}{2-a}\Big\{(r+s-r-1)^2 + 2(r+s) - 3 + (k-(r+s)+1-a)\Big(2(r+s)-r-1+\frac{k-(r+s)-a}{2}\Big)\Big\}.$$

Since $r \geqslant 1$, and replacing $r+s$ by t which now ranges in $[2, k]$,

$$n \leqslant k + \frac{2}{2-a}\Big\{(t-2)^2 + 2t - 3 + (k-t+1-a)\Big(2t-2+\frac{k-t-a}{2}\Big)\Big\}.$$

Routine manipulation now shows that if $a = 0$, then $n \leqslant k^2 + k - 1$, while if $a = 1$, then $n \leqslant 2k^2 - 3k + 2$. Since n exceeds the specified bound in each case, G is Hamiltonian.

Non-Hamiltonian graphs satisfying the conditions of regularity of degree $n-k$ with $2n$ or $2n-1$ vertices, and two connectedness, are known. For example, choose graphs H_1', H_2', and H_3' such that H_i' is isomorphic to K_{2t}. In $V(H_i')$, choose disjoint sets A_i and B_i, each of cardinality $2t/3 - [i/3]$, and form H_i from H_i' by deleting from H_i' a matching, each of whose edges joins a member of A_i to a member of B_i. Form a graph H by joining a new vertex u to every member of every A_i and a new vertex v to every member of every B_i. Then, letting $k = t + 2$ and $n = 3k - 5$, H is non-Hamiltonian, has $2n$ vertices, and is two connected and regular of degree $n - k$. Many other similar examples can be constructed. Thus the theorem clearly requires some lower bound for n. But this lower bound surely is not as large as the ones used here.

References

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