EULER'S ϕ-FUNCTION AND ITS ITERATES

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Introduction. In this paper we continue our study of the values taken by Euler's ϕ-function begun in [1]–[3]. Let $\phi_r(n)$ be the iterated ϕ-function, that is $\phi_r(n) = \phi\{\phi_{r-1}(n)\}$ where $\phi_1 = \phi$. Let

$$V_r(x) = \text{card}\,\{m < x : m = \phi_r(n) \text{ for some } n\}.$$

In [2] and [3] we obtained, respectively

$$V(x) = V_1(x) \ll \frac{x}{\log x} \exp\left(B\sqrt{\log\log x}\right),$$

$$V_1(x) \gg \frac{x}{\log x} \exp\{C(\log\log\log x)^2\},$$

and our present aim is to obtain an upper bound for $V_2(x)$. Our result is as follows.

THEOREM. *There exists an absolute constant D such that*

$$V_2(x) \ll \frac{x}{\log^2 x} \exp\left(D\,\frac{\log\log x \,.\, \log\log\log\log x}{\log\log\log x}\right).$$

Remarks. In the case $r = 1$ the simple lower bound $V_1(x) \geqslant \pi(x)$ is available. When $r = 2$, the analogous result is $V_2(x) \geqslant \pi_2(x)$ where $\pi_2(x)$ denotes the number of primes $p < x$ such that $(p-1)/2$ is prime. Evidently the numbers

$$\phi_2(p) = (p-3)/2$$

are distinct. Sieve theory suggests, but of course does not yet prove, that $\pi_2(x) \gg x/\log^2 x$, so that apart from the second factor on the right our estimate is probably sharp. We hope to return to the lower bound problem: our best result so far is $x/\log^k x$ for some fixed $k > 2$.

It may be that for every fixed r and every $\varepsilon > 0$ we have

$$x/(\log x)^{r+\varepsilon} \ll V_r(x) \ll x/(\log x)^{r-\varepsilon}.$$

Notation. $v(n)$ denotes the number of distinct prime factors of n and $\omega(n)$ the total number of prime factors. $P^+(n)$ and $P^-(n)$ denote respectively the greatest and least prime factors of n. Other notation will be made clear in the proof.

It is convenient to work with the function

$$W_2(x) = \text{card}\{m : m = \phi_2(n) \text{ for some } n < x\}.$$

This is smaller than $V_2(x)$, but since $\phi(n) \gg n/\log\log n$ we have

$$V_2(x) \leqslant W_2\big(cx(\log\log x)^2\big)$$

and this does not alter the final result.

LEMMA 1. *For each fixed A, there exists $B = B(A)$ such that*

$$\operatorname{card}\{n < x : \omega(n) > B\log\log x\} \ll x(\log x)^{-A}.$$

Proof. Choose an integer h so that $h\log h - h \geqslant A$, and let $\omega(n, h)$ denote the total number of prime factors of n greater than h. For all y,

$$(1+y)^{\omega(n,h)} = \sum_{d|n}{}' y^{\nu(d)}(1+y)^{\omega(d)-\nu(d)}$$

where the dash denotes the restriction on d that all its prime factors exceed h. This formula is most easily proved by noting that the summand on the right, and therefore the sum, is multiplicative. Hence for $0 \leqslant y < h$,

$$\sum_{n<x}(1+y)^{\omega(n,h)} \leqslant x\sum_{d<x}{}'\frac{y^{\nu(d)}}{d}(1+y)^{\omega(d)-\nu(d)}$$

$$\leqslant x\prod_{h<p<x}\left(1+\frac{y}{p-1-y}\right),$$

since $d < x$ implies that all its prime factors are less than x. This does not exceed

$$x\exp\left\{\sum_{h<p<x}\frac{y}{p-1-y}\right\} = x\exp\left\{\sum_{h<p<x}\left(\frac{y}{p}+\frac{y(1+y)}{p(p-1-y)}\right)\right\}$$

$$\ll x(\log x)^y\exp\left\{\sum_{p>h}\frac{y(1+y)}{p(p-1-y)}\right\}.$$

The sum in the exponential is convergent, and if we set $y = h-1$ we may deduce that

$$\sum_{n<x} h^{\omega(n,h)} \leqslant C(h)x(\log x)^{h-1}$$

where $C(h)$ is a function of h only. Hence

$$\operatorname{card}\{n < x\colon \omega(n,h) > h\log\log x\} \leqslant C(h)x(\log x)^{h-1-h\log h},$$

and by the definition of h, the right hand side is $\ll x(\log x)^{-A}$. Next, let

$$\omega'(n,h) = \omega(n) - \omega(n,h)$$

and suppose $\omega'(n,h) > s\log\log x$. Then n has a divisor $d = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_u^{\alpha_u}$ where $p_1, p_2, \dots, p_u$ are the primes up to h and $\alpha_1+\alpha_2+\dots+\alpha_u = l = [s\log\log x]$. Hence

$$\operatorname{card}\{n < x : \omega'(n,h) > s\log\log x\} \leqslant \sum x/p_1^{\alpha_1}p_2^{\alpha_2}\dots p_u^{\alpha_u} \leqslant \sum x/2^l,$$

where summation is over all choices of the exponents α_i such that their sum is l. There are at most $(l+1)^u$ such choices, and so if $s > A/\log 2$, the right hand side is $\ll x(\log x)^{-A}$. Now put $B = h+s$. Then $\omega(n) > B\log\log x$ implies either $\omega(n,h) > h\log\log x$ or $\omega'(n,h) > s\log\log x$, and we obtain the result stated.

LEMMA 2. *Let $A > 0$ and $u^2 > 2v > 0$. Then*

$$\operatorname{card}\{n < x : \nu(n) \geqslant u, \omega\{\phi(n)\} \leqslant v\}$$

$$\ll x(\log x)^{-A} + \frac{u^2 x\log\log x}{v\log x}\exp\{2u - u\log(u^2/2v)\}.$$

Proof. Let $y > 1 > z > 0$. Then the cardinality in question does not exceed

$$y^{-u} z^{-v} \sum_{\sqrt{x}<n<x}' y^{v(n)} z^{\omega\phi(n)} + O\big(x(\log x)^{-A}\big),$$

where the dash denotes that $\omega(n) \leqslant B(A) \log\log x$, moreover in the exponential $\omega\phi(n)$ is simply a condensed form of $\omega\{\phi(n)\}$. To see this, notice that integers $n \leqslant \sqrt{x}$, or such that $\omega(n) > B(A) \log\log x$ are covered by the error term: for the remaining integers the powers of y and z are respectively $\geqslant 0$, $\leqslant 0$, which gives the inequality as $0 < z < 1 < y$. Next, we deduce that for the integers n counted by $\sum'$, we must have $P^+(n) > x^{1/2B \log\log x}$. Hence

$$\sum_{\sqrt{x}<n<x}' y^{v(n)} z^{\omega\phi(n)} \leqslant \sum_{n<x}' y^{v(n)} z^{\omega\phi(n)} \sum_{p|n}'' 1$$
$$\leqslant \sum_{p<x}'' y \sum_{m<x/p} y^{v(m)} z^{\omega\phi(m)},$$

where $''$ denotes $p > x^{1/2B \log\log x}$, that is, $\log(x/m) > (\log x)/2B \log\log x$. Therefore the left hand side does not exceed

$$y \sum_m y^{v(m)} z^{\omega\phi(m)} \pi(x/m) \ll \frac{yx(\log\log x)}{\log x} \sum_m \frac{y^{v(m)} z^{\omega\phi(m)}}{m}$$
$$\ll \frac{yx(\log\log x)}{\log x} \prod_{p<x} \left(1 + \frac{yz^{\omega(p-1)}}{p-z}\right) \ll \frac{yx(\log\log x)}{\log x} \exp\left(\frac{2yz}{1-z}\right),$$

by virtue of the estimate for $\sum p^{-1} z^{\omega(p-1)}$ in **[2]**.

Hence

$$y^{-u} z^{-v} \sum_{\sqrt{x}<m<c}' y^{v(n)} z^{\omega\phi(n)} \ll \frac{yx \log\log x}{\log x} \exp\left(\frac{2yz}{1-z} - u \log y + v \frac{1-z}{z}\right),$$

and we choose $z < 1$ such that $z/(1-z) = \sqrt{(v/2y)}$, and $y = u^2/2v$. This gives the result stated.

LEMMA 3. *Let A, $B > 0$ be arbitrary but fixed. There exists $C = C(A, B)$ such that*

$$\text{card}\left\{n < x : v(n) \geqslant \frac{C \log\log x}{\log\log\log x}, \omega\{\phi(n)\} \leqslant B \log\log x\right\} \ll x(\log x)^{-A}.$$

This follows immediately from Lemma 2. Notice that for all n, $\omega\{\phi(n)\} \geqslant \omega(n)$ so that we get the same result if we demand that $\omega\{\phi_2(n)\} \leqslant B \log\log x$.

Proof of the theorem. Let us set $A = 2$, $B = B(2)$, $C = C\big(2, B(2)\big)$ in Lemmas 1–3. Plainly we may neglect any set of integers of cardinality $\ll x \log^{-2} x$, hence in view of our lemmas we may restrict our attention to integers n satisfying the following conditions.

(i) $\omega\{\phi_2(n)\} \leqslant B \log\log x$,

(ii) $v(n) \leqslant u = \dfrac{C \log\log x}{\log\log\log x}$.

Let $m = \phi(n')$, $n' = \phi(n)$. By condition (i), $\omega(m) \leqslant B \log\log x$, hence $\omega(n') \leqslant B \log\log x$. Either $n' \leqslant \sqrt{x}$, in which case it may be neglected, or we may deduce

(iii) $P^+(n') > x^{1/2B \log\log x} = t$ say.

Hence we have

$$W_2(x) \leqslant \sum_{k \leqslant u} W(x,k) + O\big(x/(\log x)^2\big),$$

where

$$W(x,k) = \operatorname{card}\{n < x : \nu(n) = k, \omega\{\phi_2(n)\} \leqslant B\log\log x,$$
$$P^+(n') > x^{1/2B\log\log x}\}.$$

We write $\psi(n) = \omega\{\phi_2(n)\}$ and

$$S_k(x,z) = \sum z^{\psi(n)} \qquad (0 < z < 1),$$

where the sum is over the n's counted by $W(x,k)$. If $p|n$ then $\phi_2(p^\alpha n) = p^\alpha \phi_2(n)$ or $\phi(p^\alpha)\phi_2(n)$ according as $p|\phi(n)$ or not. In either case, $\psi(p^\alpha n) \geqslant \alpha + \psi(n)$. Hence

$$\sum \left\{ z^{\psi(n)} : \prod_{p|n} p = n_0 \right\} \leqslant (1-z)^{-\nu(n_0)} z^{\psi(n_0)},$$

and so

$$S_k(x,z) \leqslant (1-z)^{-k} \sum |\mu(n)| z^{\psi(n)}$$

(where we have replaced n_0 by n in the sum on the right). Suppose that

$$n = p_1 p_2 \dots p_k,\ p_1 < p_2 < \dots < p_k.$$

By (iii) above we may write $\phi(n) = n' = qr$ where r is a prime exceeding t. We drop the condition that $p_1, p_2, \dots, p_k$ should be ordered, and assume further that $r|(p_1 - 1)$. Then we have

$$S_k(x,z) \leqslant \frac{(1-z)^{-k}}{(k-1)!} \sum_{qr<x} z^{\omega\phi(qr)},$$

where $qr = (p_1 - 1)(p_2 - 1)\dots(p_k - 1)$, $p_1 - 1 = rd$, $d|q$, and as before $\omega\phi$ in the exponent means $\omega\{\phi\}$. The sum on the right does not exceed

$$\sum_{q<x/t} z^{\omega\phi(q)} \sum_{d|q}{}' \sum_{r<x/q}{}'' 1,$$

where $\sum'$ denotes that q/d is of the form $(p_2 - 1)(p_3 - 1)\dots(p_k - 1)$, and $\sum''$ that both r, and $rd + 1$, should be prime. This is

$$\ll \sum_{q<x/t} z^{\omega\phi(q)} \sum_{d|q}{}' \frac{dx/q}{\phi(d)\log x/q}$$

$$\ll \frac{x}{\log^2 t} \sum_{q<x} \frac{z^{\omega\phi(q)}}{\phi(q)} \sum_{a|q} 1,$$

where $a(= q/d) = (p_2 - 1)(p_3 - 1)\dots(p_k - 1)$. Since $\phi(q) \gg q/\log\log q$, and in view of the definition of t, this is

$$\ll \frac{x(\log\log x)^3}{\log^2 x} \sum_{a<x} \sum_{m} \frac{z^{\omega\phi(ma)}}{ma} \qquad (ma = q)$$

$$\ll \frac{x(\log\log x)^3}{\log^2 x} \left(\sum_{a<x} \frac{1}{a}\right) \sum_{m=1}^{\infty} \frac{z^{\omega\phi(m)}}{m}.$$

The sum over m (with no restriction) was estimated in **[2]**, and that over a does not exceed

$$\left(\sum_{p<x} \frac{1}{p-1}\right)^{k-1}.$$

Hence $W(x, k) \leqslant z^{-B\log\log x} S_k(x, z)$ and

$$S_k(x, z) \ll \frac{x(1-z)^{-k}(\log\log x + O(1))^{k+2}}{(k-1)!\log^2 x} \exp\left(\frac{2z}{1-z}\right).$$

We choose z so that $z/(1-z) = \log\log\log x$. Since $k \leqslant u$, we deduce from Stirling's formula that

$$S_k(x, z) \ll \frac{x}{\log^2 x} \exp\left(O\left(\frac{\log\log x . \log\log\log\log x}{\log\log\log x}\right)\right),$$

and hence this estimate is true of $W_2(x)$, and $V_2(x)$, as required.

References

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