# SETS OF INDEPENDENT EDGES OF A HYPERGRAPH

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GIVEN a set $X$ and a natural number $r$ denote by $X^{(r)}$ the set of $r$-element subsets of $X$. An $r$-graph or hypergraph $G$ is a pair $(V, T)$, where $V$ is a finite set and $T \subset V^{(r)}$. We call $v \in V$ a *vertex* of $G$ and $\tau \in T$ an $r$-tuple or an edge of $G$. Thus a 1-graph is a set $V$ and a subset $T$ of $V$. As the structure of 1-graphs is trivial, throughout the note *we suppose* $r \geqslant 2$. A 2-graph is a graph in the sense of (**5**). The *degree* deg $v$ of a vertex $v \in V$ is the number of $r$-tuples containing $v$. A set of pairwise disjoint $r$-tuples is said to be independent. We say $G' = (V', T')$ is a subgraph of $G = (V, T)$ and write $G' \subset G$ if $V' \subset V$ and $T' \subset T$. If $G = (V, T)$ and $v \in V$ then $G - v = (V', T')$, where $V' = V - \{v\}$ and $T' = \{\tau \in T : v \notin \tau\}$. If $X$, $Y$ are sets $|X|$ denotes the cardinality of $X$ and $X - Y$ is the set theoretic difference of $X$ and $Y$. An $r$-graph with $p$ vertices and all $\binom{p}{r}$ possible $r$-tuples is denoted by $K_p$. Thus $K_p$ is the complete graph with $p$ vertices. Also $\bar{K}_p$ is the graph with $p$ vertices and no $r$-tuples.

Let $E_r(n, k)$ $(0 \leqslant k \leqslant n)$ be an $r$-graph $(V, T)$, where $|V| = n$ and $T = \{\tau \in V^{(r)} : \tau \cap W \neq \phi\}$ for some $k$-element subset $W$ of $V$. (Thus adapting the notation of (**5**) to $r$-graphs, $E_r(n, k) = K_k + \bar{K}_{n-k}$.) Put

$$e_r(n, k) = |T| = \binom{n}{r} - \binom{n-k}{r}.$$

The graph $E_r(n, k)$ clearly does not contain $k+1$ independent $r$-tuples and it is *maximal* with this property if $n \geqslant (k+1)r$. Let us define another maximal $r$-graph with at most $k$ independent $r$-tuples, $F_r(n, k) = (V_1, T_1)$. Let $|V_1| = n \geqslant k+r$, let $W_1$ and $R$ be disjoint subsets of $V_1$, $|W_1| = k-1$, $|R| = r$, and let $v \in V_1 - W_1 - R$. Then the set of $r$-tuples of $F_r(n, k)$ is

$$T_1 = \{\tau \in V_1^{(r)} : \tau \cap W_1 \neq \phi\} \cup \{\tau \in V_1^{(r)} : v \in \tau \text{ and } \tau \cap R \neq \phi\} \cup \{R\}.$$

If $n \geqslant (k+1)$ then $F_r(n, k)$ is a maximal $r$-graph without $k+1$ independent $r$-tuples. Put

$$f_r(n, k) = |T_1| = \binom{n}{r} - \binom{n-k}{r} - \binom{n-k-r}{r-1} + 1$$
$$= e_r(n, k) - \binom{n-k-r}{r-1} + 1.$$

It was proved by Erdös and Gallai [(**3**) theorem 4.1] that if a 2-graph $G$ on $n[> (5k+3)/2]$ vertices has at least $e_2(n, k)$ edges and does not contain $k+1$ independent edges then $G$ is exactly $E_2(n, k)$. This result was extended to $r$-graphs by Erdös (**2**) in the following form.

Given $r \geqslant 2$ there exists a constant $c_r$ such that every $r$-graph with $n > c_r k$ vertices and $e_r(n, k)+1$ or more $r$-tuples contains $k+1$ independent $r$-tuples. The proof of this result is based on the corresponding theorem for $k = 1$ and arbitrary $r$, proved by Erdös, Ko and Rado (**4**). It is conjectured in (**2**) that if an $r$-graph with $n \geqslant (k+1)r$ vertices contains more than

$$\max\left[\left(\tbinom{(k+1)r-1}{r}\right), e_r(n, k)\right]$$

$r$-tuples then it contains $k+1$ independent $r$-tuples. This conjecture is still open for all $r \leqslant 3$.

Sharpening the result of Erdös, Ko and Rado (**4**) it was proved by Hilton and Milner (**7**) that if an $r$-graph without 2 independent $r$-tuples has $n \geqslant 2r$ vertices and $f_r(n, 1)+1$ or more $r$-tuples then it is a subgraph of $E_r(n, 1)$.

In this note we sharpen the result of Erdös (**2**) (and put it in a more explicit form) by extending the result of Hilton and Milner (**7**) for every $k \geqslant 1$ (Theorem 1), provided $n > 2r^3k$. Naturally the graph $F_r(n, k)$ shows that fewer $r$-tuples do not imply the assertion. An immediate consequence of Theorem 1 is an extension of a result of Hilton (**6**) concerning sets of independent $r$-tuples (Corollary 1).

The main aim of this note is to give another condition on an $r$-graph $G$ that ensures $k+1$ independent $r$-tuples unless $G \subset E_r(n, k)$. Instead of requring a sufficient number of $r$-tuples, we require that the *degree of each vertex be sufficiently large* (Theorem 2).

The minimal degree in $E_r(n, k)$ is

$$\binom{n-1}{r-1} - \binom{n-k-1}{r-1} = e_{r-1}(n-1, k).$$

It follows from Theorem 2 that if in an $r$-graph $G$ on $n[> 2r^3(k+2)]$

vertices the degree of every vertex is greater than the above then $G$ contains $k+1$ independent $r$-tuples. The graph $E_r(n, k)$ shows that this condition on the degrees can not be weakened if we want to ensure the existence of $k+1$ independent $r$-tuples.

It is interesting to note that the graph $E_r(n, k)$ is also the unique solution of the following extremal problem. An $r$-graph $H$ is said to be $(r+k)$-*saturated* if $H$ is a maximal $r$-graph which does not contain a $K_{r+k}$. Then among $(r+k)$-saturated $r$-graphs on $n(\geqslant r+k)$ vertices $E_r(n, k)$ is the unique graph with the minimal number of $r$-tuples. This was proved by Bollobás in (**1**) using the method of weights.

In the proofs of our theorems, we shall make use of the following simple inequalities.

$$l\binom{m-1}{s-1} \geqslant \binom{m}{s} - \binom{m-l}{s} \geqslant l\binom{m-l}{s-1}, \qquad \text{where } 1 \leqslant s \leqslant m-l \leqslant m. \quad (1)$$

$$\binom{m-l}{s}\Big/\binom{m}{s} \geqslant \left(1-\frac{l}{m-s}\right)^s \geqslant 1-\frac{sl}{m-s}, \qquad \text{where } 0 \leqslant \delta < m-l \leqslant m. \quad (2)$$

[The second inequality of (2) follows from $(1-x)^s \geqslant 1-sx$ if $0 \leqslant x < 1$.]

We shall also make use of the following simple lemma whose proof we omit [cf. the proof in (**2**)].

LEMMA 1. *Let* $G = (V, T)$ *be an* $r$*-graph on* $n$ *vertices containing at most* $p \geqslant 1$ *independent* $r$*-tuples.*

(a) *If* $u \in V$ *and* $G-u$ *contains* $p$ *independent* $r$*-tuples then*

$$\deg u \leqslant \binom{n-1}{r-1} - \binom{n-1-rp}{r-1} \leqslant rp\binom{n-2}{r-2}.$$

(b) *There is a vertex* $v$ *in* $G$ *such that*

$$\deg v \geqslant \frac{|T|}{rp}.$$

THEOREM 1. *Let* $G = (V, T)$ *be an* $r$*-graph with*

$$r \geqslant 2,\ k \geqslant 1,\ |V| = n > 2r^3k \text{ and } |T| > f_r(n, k).$$

*Suppose* $G$ *contains at most* $k$ *independent* $r$*-tuples. Then* $G \subset E_r(n, k)$*; in other words there exists* $W \subset V$ *with* $|W| = k$ *such that every* $r$*-tuple of* $G$ *intersects* $W$*.*

*Proof.* For $k = 1$ this was proved by Hilton and Milner (**7**), so suppose $k > 1$ and that the result holds for smaller values of $k$.

Suppose first that there is a vertex $u \in V$ such that $G-u$ has at most $k-1$ independent $r$-tuples. As

$$|T| - \deg u > f_r(n, k) - \deg u \geqslant f_r(n, k) - \binom{n-1}{r-1} = f_r(n-1, k-1),$$

the induction hypothesis implies that $G-u \subset E_r(n-1, k-1)$ and so $G \subset E_r(n, k)$.

Suppose now that $G-u$ has $k$ independent $r$-tuples for every vertex $u \in V$.

The two parts of Lemma 1 imply that if $G$ is not a subgraph of $E_r(n, k)$ then

$$\frac{|T|}{rk} \leqslant rk\binom{n-2}{r-2}. \tag{3}$$

By (1) we have

$$|T| > f_r(n, k) \geqslant k\binom{n-k}{r-1} - \binom{n-k-r}{r-1} + 1 > (k+1)\binom{n-k}{r-1},$$

and it follows from (3) and (2) that

$$(rk)^2 \geqslant \frac{(k-1)(n-k)}{r-1}\left\{1 - \frac{(r-2)(k-1)}{n-r}\right\}.$$

Routine calculations show that this contradicts the assumption $2r^3k < n$, and the proof is complete.

LEMMA 2. *Let $F = (V, T)$ be an $r$-graph with*

$$r \geqslant 2,\ k \geqslant 2,\ |V| = n > 2r^3(k-1) \text{ and } |T| \geqslant f_r(n, k-1).$$

*Suppose every $r$-tuple of $F$ meets a set $W$ having $|W| = k-1$. Let $\tau$ be an $r$-tuple which does not meet $W$. Then the $r$-graph $F \cup \tau$ has $k$ independent $r$-tuples.*

*Proof.* The number of $r$-tuples of $F$ which meet $\tau$ is at most

$$h = \binom{n}{r} - \binom{n-r}{r} - \binom{n-k+1}{r} + \binom{n-r-k+1}{r}.$$

The case $k = 2$ follows because $h < f_r(n, k-1)$, so we assume $k \geqslant 3$. The number $h'$ of $r$-tuples of $F - \tau$ satisfies

$$h' \geqslant f_r(n, k-1) - h = 1 + e_r(n-r, k-2) > f_r(n-r, k-2).$$

If $F-\tau$ has $k-1$ independent $r$-tuples then those together with $\tau$ give the desired result. Suppose on the other hand $F-\tau$ has at most $k-2$ independent $r$-tuples. Then by Theorem 1 we know $F-\tau \subset E_r(n-r, k-2)$ so $h' \leqslant e_r(n-r, k-2)$. This contradiction completes the proof.

To formulate the next result let us recall a definition of Hilton (**6**). We say that an $r$-graph $G$ contains a *simultaneously independent k-sets* if there are $sk$ of the $r$-tuples that can be partitioned into $s$ classes, such that each class contains $k$ independent $r$-tuples.

COROLLARY 1. *Let* $G = (V, T)$ *be an r-graph with*

$$r \geqslant 2, k \geqslant 2, |V| = n > 2r^3(k-1)$$

*and*

$$|T| \geqslant f_r(n, k-1)+(s+1)k-1.$$

*Suppose G has at most s simultaneously independent k-sets. Then there are s of the r-tuples of G such that the r-graph obtained from G by omitting these r-tuples is a subgraph of an* $E_r(n, k-1)$.

*Proof.* Let $p$ be the largest integer for which $G$ has $p$ simultaneously independent $k$-sets and let $S$ denote such a family. If $G' = (V, T')$ where $T' = T-S$ then by definition of $p$ there are at most $k-1$ independent $r$-tuples in $G'$. Since

$$|T'| = |T|-pk \geqslant f_r(n, k-1)+k-1,$$

by Theorem 1 there is a set $W$ with $|W| = k-1$ such that every $r$-tuple of $G'$ meets $W$. Now each class of $S$ must contain an $r$-tuple which fails to meet $W$, but suppose some class $C$ contained two such $r$-tuples $\tau$ and $\sigma$. Then Lemma 2 shows that $G' \cup \tau$ has $k$ independent $r$-tuples and we will denote them by $C_1$. If we omit $C_1$ from $G' \cup \tau$ and adjoin $\sigma$ we can again apply the lemma to get a second set $C_2$ of $k$ independent $r$-tuples. However replacing $C$ in $S$ by $C_1$ and $C_2$ contradicts the definition of $p$. Thus we have shown that each class of $S$ contains exactly one $r$-tuple which fails to intersect $W$ and omitting these $r$-tuples from $G$ produces a subgraph of $E_r(n, k-1)$.

It is likely that a somewhat more careful proof would show that the same assertion holds if we require only that $|T| \geqslant f_r(n, k-1)+s$.

For the next theorem and its corollary notice that

$$\binom{n-1}{r-1} - \binom{n-k}{r-1}$$

is the minimum degree in $E_r(n, k-1)$.

THEOREM 2. *Let $G = (V, T)$ be an r-graph with*

$$r \geqslant 2, k \geqslant 1 \text{ and } |V| = n > 2r^3(k+2).$$

*Suppose G contains at most k independent r-tuples. If*

$$\deg v > d = d_r(n, k) = \binom{n-1}{r-1} - \binom{n-k}{r-1} + \frac{r^3}{n-k+1}\binom{n-k-1}{r-2}$$

*for every $v \in V$ then $G \subset E_r(n, k)$.*

*Proof.* We shall prove the theorem by induction on $k$. Suppose first that $k = 1$. By Lemma 1b there is a vertex $v$ such that

$$\deg v \geqslant |T|/r > nd/r^2 = r\binom{n-2}{r-2}.$$

Let $H = G - v$. Then $H$ can not have an $r$-tuple since otherwise Lemma 1a contradicts the previous inequality. Thus every $r$-tuple of $G$ contains $v$ and so $G \subset E_r(n, 1)$.

Suppose now that $k > 1$ and the result holds for smaller values of $k$. As in the case $k = 1$, Lemma 1b implies that there exists a vertex $v$ such that

$$\deg v \geqslant \frac{|T|}{rk} > \frac{nd}{r^2k}. \tag{4}$$

Put $H = G - v$. Then

$$\deg_H u \geqslant \deg_G u - \binom{n-2}{r-2} > d_r(n, k) - \binom{n-2}{r-2} = d_r(n-1, k-1)$$

for every vertex $u$ of $H$.

If $H$ contains at most $k-1$ independent $r$-tuples, the induction hypothesis implies that there is a set $W$ with $|W| = k-1$ such that every $r$-tuple in $H$ meets $W$. Hence in this case every $r$-tuple of $G$ meets $W \cup \{v\}$ and $|W \cup \{v\}| = k$.

Thus we can assume without loss of generality that $H$ contains $k$ independent $r$-tuples. Then by Lemma 1a we have

$$\deg v \leqslant rk\binom{n-2}{r-2}.$$

Consequently (4) gives

$$\frac{r^3k^2}{n}\binom{n-2}{r-2} > d > \binom{n-1}{r-1} - \binom{n-k}{r-1}.$$

Thus (1) implies

$$\frac{r^3k^2}{n}\binom{n-2}{r-2} > (k-1)\binom{n-k}{r-2},$$

and so by (2)

$$\tfrac{1}{2} > \frac{r^3k^2}{n(k-1)} > \binom{n-k}{r-2}\Big/\binom{n-2}{r-2} \geqslant 1-\frac{(r-2)(k-2)}{n-r}.$$

This contradicts our assumption on $n$, so the theorem is proved.

Notice that the number of $r$-tuples in $G$ guaranteed by the condition on the degrees is less than $f_r(n, k)$ so Theorem 2 does not follow directly from Theorem 1.

Corollary 2. *Let $G = (V, T)$ be an $r$-graph with*

$$r \geqslant 2,\ k \geqslant 2 \text{ and } |V| = n > 2r^3(k+1).$$

*Suppose that*

$$1 \leqslant s \leqslant \tfrac{1}{2}\binom{n-k}{r-2}$$

*and*

$$\deg v > \binom{n-1}{r-1} + \frac{rk(s-1)}{n-k+1}$$

*for every $v \in V$. Then $G$ has $s$ simultaneously independent $k$-sets.*

*Proof.* Let $p$ be the largest integer for which $G$ has $p$ simultaneously independent $k$-sets and let $S$ denote such a family. We assume $p < s$ and obtain a contradiction. If $G' = (V, T')$ where $T' = T - S$ then there are at most $k-1$ independent $r$-tuples in $G'$, and for every $v \in V$

$$\deg_{G'} v \geqslant \deg_G v - p > d_r(n, k-1).$$

Hence by Theorem 2 there is a set $W$ with $|W| = k-1$ such that every $r$-tuple of $G'$ meets $W$. Clearly

$$\deg_{G'} z \geqslant \binom{n-1}{r-1} - \binom{n-k}{r-1}$$

for every $z \in V - W$, but there is at least one $z_0 \in V - W$ for which

$$\deg_G z_0 \leqslant \deg_{G'} z_0 + \frac{rkp}{n-k+1}$$

contradicting our hypothesis about $\deg v$.

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It is easily seen that the restrictions on the parameters in Theorem 2 and Corollary 2 can be weakened by proving a more accurate result for $k = 2$.

We are grateful to Dr. A. J. W. Hilton for drawing our attention to (**6**).

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