ON THE NUMBER OF DISTINCT PRIME DIVISORS OF $\binom{n}{k}$

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Denote by $V(n,k)$ the number of distinct prime divisors of $\binom{n}{k}$. It is well known and easy to see that for $n > n_0(k)$, $V(n,k) \geq k$ and it is very likely that $V(n,k) = k$ for infinitely many n. Denote by m_k the least positive integer n for which $V(n,k) = k$, by n_k the least one for which $V(n,k) \geq k$ and by N_k the smallest integer so that for every $n \geq N_k$, $V(n,k) \geq k$.

We have tabulated the complete factorizations of $\binom{n}{k}$ for $n \leq 551$, $k \leq 25$. We have thus obtained values of m_k for $k \leq 25$. We cannot, however, prove that m_k exists for all k. It is interesting to note that m_k is not always less than m_{k+1}. Thus for example $m_{17} > m_{19} > m_{18}$, also $m_{51} > n_{51}$ and $m_{28} > m_{26} > m_{27}$. On the basis of our table one would guess that $n_k < k^2$ always holds. In fact, we shall prove that this conjecture completely fails. We have

THEOREM 1. *$n_k > k^2$ for $k > 4939$. Further, for every $\varepsilon > 0$ there is a $k_0(\varepsilon)$ so that for all $k > k_0(\varepsilon), n_k > (1-\varepsilon)\, k^2 \log k$.*

With somewhat longer computation we could determine all the integers k with $n_k \leq k^2$. It seems certain that

$$\lim_{k=\infty} \frac{n_k}{k^2 \log k} = \infty$$

is true and perhaps its proof is not too difficult, though we have succeeded in proving it.

THEOREM 2. $\limsup_{k=\infty} \dfrac{\log n_k}{\log k} \leq e, \quad \liminf_{k=\infty} \dfrac{\log N_k}{\log k} \geq e$.

It seems very difficult to get a good upper bound for N_k. Here we prove

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THEOREM 3. *For every* $\varepsilon > 0$, $k > k_0(\varepsilon)$, $N_k < (e+\varepsilon)^k$.

P. Erdös has stated this without proof in [1]. P. Erdös and E. Szemeredi (unpublished) proved in fact a slightly stronger result: there is an $\alpha < e$ such that $N_k < \alpha^k$ for $k > k_0$.

$$\lim_{k=\infty} N_k^{1/k} = 1$$

certainly holds but we can not prove it.

Proof of Theorem 1. Let $2 = p_1 < p_2 < \ldots$ be the sequence of consecutive primes. A theorem of Rosser [2] states that for every j, $p_j > j \log j$. Thus by Stirling's formula, we obtain

$$(1) \quad \binom{n_k}{k} \geq \prod_{i=1}^{k} p_i \geq \prod_{t=2}^{k} t \log t = k! \prod_{t=2}^{k} \log t > k^k e^{-k} \prod_{t=2}^{k} \log t .$$

On the other hand, if $n_k \leq k^2$, we evidently have

$$(2) \qquad \binom{n_k}{k} < \frac{n_k^k}{k!} \leq \frac{k^{2k} e^k}{k^k} = k^k e^k .$$

Now (1) and (2) imply that

$$\prod_{t=2}^{k} \log t < e^{2k} ,$$

or what is the same thing

$$\sum_{t=2}^{k} \log \log t < 2k .$$

This is false for $k > 4939$, thus for $k > 4939, n_k > k^2$. Further, for $k > k_0(\varepsilon)$ we obtain by a simple computation

$\sum_{t=2}^{k} \log \log t > 2k + (1-\varepsilon) k \log \log k$. Thus from (1) and (2) we easily obtain that for $k > k_0(\varepsilon), n_k > (1-\varepsilon) k^2 \log k$, which completes the proof of Theorem 1.

Proof of Theorem 2. First we prove that for every $\alpha > 1$ and $k \to \infty$

$$(3) \qquad \sum_{n=k}^{k^\alpha} V(n,k) = (1+o(1))k^{1+\alpha} \log \alpha .$$

To prove (3) observe that if p is any prime greater than k then $p \mid \binom{n}{k}$ if and only if $p \mid (n-j)$ for some j, $0 \le j < k$. Thus we evidently have

$$(4) \qquad \sum_{n=k}^{k^\alpha} V(n,k) = \sum_{k \le p \le k^\alpha} k \frac{k^\alpha}{p} + O(k^\alpha \pi(k)) + O(k\pi(k^\alpha)) .$$

The first error term in (4) is contributed by the primes not exceeding k and the second by the primes $k < p \le k^\alpha$. From (4) we obtain (3) from $\pi(k) = o(k)$ and the well known theorem of Mertens

$$\sum_{k<p<k^\alpha} \frac{1}{p} = \log \alpha + o(1) .$$

From (3) we obtain that for $k > k_0(\varepsilon)$,

$$(5) \qquad \frac{1}{k^{e+\varepsilon}-k} \sum_{n=k}^{k^{e+\varepsilon}} V(n,k) > 1$$

and

$$(6) \qquad \frac{1}{k^{e-\varepsilon}-k} \sum_{n=k}^{k^{e-\varepsilon}} V(n,k) < 1-\eta,\ \eta = \eta(\varepsilon) .$$

(5) implies that, for some $n \le k^{\alpha+\varepsilon}$, $V(n,k) > k$ or $n_k < k^{e+\varepsilon}$, and (6) implies that, for some $n > k^{e+\varepsilon}$, $V(n,k) < k$ or $N_k > k^{e-2\varepsilon}$ which proves theorem 2.

One is tempted to conjecture

$$(7) \qquad \lim_{k=\infty} \frac{\log n_k}{\log k} = \lim_{k=\infty} \frac{\log N_k}{\log k} = e ,$$

but if (7) is true it must be very deep. As a modest step towards the proof of (7) we conjecture

$$\sum_{n=k}^{k^{\alpha}} V(n,k)^2 = (1 + o(1))\, k^{2+\alpha}(\log \alpha)^2 . \tag{8}$$

(8) would imply that for all but $o(k^{\alpha})$ integers $n < k^{\alpha}$, $V(n,k) = (1 + o(1))\, k \log \alpha$.

Proof of Theorem 3. We say the prime p belongs to $(n-i)$, $0 \le i < k$, if $p^{\alpha} \| (n-i)$, $p^{\alpha} > k$ holds. It is easy to see that if p belongs to $(n-i)$, then $p \mid \binom{n}{k}$. Observe further that a prime p can belong to at most one integer $(n-i)$, $0 \le i < k$. Clearly if for every i, $0 \le i < k$, at least one prime belongs to $n-i$, we obtain $V(n,k) \ge k$. The theorem now follows from the

LEMMA. *To every* $\varepsilon > 0$, *there is a* $k_0(\varepsilon)$ *so that for every* $k > k_0(\varepsilon)$ *and* $n > (e+\varepsilon)^k$ *at least one prime belongs to* $n-i$ *for every* i, $0 \le i < k$.

Assume that no prime belongs to some $n-i$, $0 \le i < k$. Let $n - i = \prod p_h^{a_h}$ be the canonical decomposition of $(n-i)$ as a product of primes. Then since each of the factors in the expression is less than or equal to k, we must have

$$n - i \le k^{\pi(k)} = e^{\pi(k) \log k} = e^{(1 + o(1))k} ,$$

an evident contradiction. Thus our lemma and the theorem are proved.

On the basis of our tables, we can now state that

$$N_2 = 4, \quad N_3 = 9, \quad N_4 = 15, \quad N_5 \ge 33, \quad N_6 \ge 63,$$
$$N_7 \ge 88, \quad N_8 \ge 170, \; N_9 \ge 133;$$

and with a little more computation we could easily determine N_k for small values of k.

By the way, it seems certain that for $2 \le k \le n/2$, $\binom{n}{k}$ is the product of consecutive primes only for a finite number of values of n and k, but we can not even prove that

$$\binom{n}{2} = \prod_{i=1}^{k} p_i$$

has only a finite number of solutions; $n = 21$ is probably the largest such n.

It seems certain that for every k there are infinitely many integers n for which $\binom{n}{i}$, $1 \le i \le k$ is the product of i distinct primes.

In the tables that follow, we list some interesting facts of this type besides giving the complete factorizations of $\binom{n_k}{k}$ for $k \le 25$. Within the limits of our table $\binom{378}{22}$ is the only one which is divisible by each of the first 13 primes.

REFERENCES

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Table 1

Complete factorization of $\binom{n_k}{k}$, $1 \le k \le 25$.

k	n	$\binom{n_k}{k}$
1	2	2
2	4	2.3
3	9	2^2.3.7
4	10	2.3.5.7
5	22	2.3^2.7.11.19
6	26	2.5.7.11.13.23
7	40	2^3.3.5.13.17.19.37
8	50	2.3.5^2.7.11.23.43.47
9	54	2^2.3.5.7.13.17.23.47.53
10	55	2.3.5.7.11.13.17.23.47.53
11	78	2^2.3.5.7.13.17.19.23.37.71.73
12	115	2^2.3.5.7.13.19.23.37.53.107.109.113
13	123	2^2.3.7.11.17.19.23.29.37.41.59.61.113
14	154	2^2.3^2.5.7.11.17.19.29.37.47.71.73.149.151
15	155	2^2.3.5.7.11.17.19.29.31.37.47.71.73.149.151
16	209	3.5.7.11.13.17.19.23.29.41.67.97.101.103.197.199
17	288	2^5.3^2.5.7.11.13.19.23.31.41.47.71.137.139.277.281.283
18	220	2.3.5.7.11.19.23.29.31.41.43.53.71.73.103.107.109.211
19	221	2.3.5.7.11.13.17.23.29.31.41.43.53.71.73.103.107.109.211
20	292	2.3.5.7.11.13.17.23.29.31.41.47.71.73.97.137.139.277.281.283
21	301	2.3.5.7.11.13.17.23.29.37.41.43.47.59.71.73.97.149.281.283.293
22	378	2.3^3.5^2.7.11.13.17.19.23.29.31.37.41.47.53.61.73.179.181.359 .367.373
23	494	2^4.3^3.5.7.11.13.17.19.29.37.41.43.53.59.61.79.97.163.239.241 .479.487.491
24	494	2.3^3.5.7.11.13.17.19.29.37.41.43.53.59.61.79.97.157.163.239 .241.479.487.491
25	551	2^2.3^2.7.11.13.17.19.23.29.31.41.53.59.61.67.89.107.109.137.179 .181.269.271.541.547

Table 2

Factorization of $\binom{n}{25}$ where the factors are distinct primes.

n	
26	2.13
43	2.3.7.13.19.29.31.37.41.43
61	3.7.13.19.29.37.41.43.47.53.59.61
62	2.3.7.13.19.29.31.41.43.47.53.59.61
125	3.5.11.13.17.29.31.37.41.53.59.61.101.103.107.109.113
223	3.13.17.29.31.37.41.43.53.67.71.73.101.103.107.109.199.211.223
233	2.3.7.11.19.29.31.37.43.53.71.73.107.109.113.211.223.227.229.233
286	2.3.11.13.19.31.47.53.67.71.89.131.137.139.263.269.271.277.281.283
287	3.7.11.13.19.31.41.47.53.67.71.89.137.139.263.269.271.277.281.283
314	2.3.7.13.29.31.37.43.59.61.73.97.101.103.149.151.157.293.307.311.313
377	5.11.13.17.19.29.31.37.41.47.53.59.61.71.73.89.179.181.353.359.367 .373
431	2.11.13.17.37.41.43.47.53.59.61.71.83.103.107.137.139.211.409.419 .421.431
475	3.11.13.19.29.31.41.43.47.59.67.79.113.151.157.227.229.233.457 .461.463.467
538	2.13.23.29.31.37.41.43.47.53.59.67.89.103.107.131.173.179.257 .263.269.521.523

Table 3

Factorization of $\binom{23}{k}$, $1 \le k \le 11$ and $\binom{47}{k}$, $1 \le k \le 20$ which are all products of distinct primes.

k	n = 23	k	n = 47
1	23	1	47
2	11.23	2	23.47
3	7.11.23	3	3.5.23.47
4	5.7.11.23	4	3.5.11.23.47
5	7.11.19.23	5	3.11.23.43.47
6	3.7.11.19.23	6	3.7.11.23.43.47
7	3.11.17.19.23	7	3.11.23.41.43.47
8	2.3.11.17.19.23	8	3.5.11.23.41.43.47
9	2.5.11.17.19.23	9	5.11.13.23.41.43.47
10	2.7.11.17.19.23	10	11.13.19.23.41.43.47
11	2.7.13.17.19.23	11	13.19.23.37.41.43.47
		12	3.13.19.23.37.41.43.47
		13	3.5.7.19.23.37.41.43.47
		14	3.5.17.19.23.37.41.43.47
		15	3.11.17.19.23.37.41.43.47
		16	2.3.11.17.19.23.37.41.43.47
		17	2.3.11.19.23.31.37.41.43.47
		18	2.5.11.19.23.31.37.41.43.47
		19	2.5.11.23.29.31.37.41.43.47
		20	2.7.11.23.29.31.37.41.43.47

Table 4

Solutions of $\binom{n}{k}$ = product of consecutive primes.

$\binom{4}{2} = 2.3$	$\binom{14}{4} = 7.11.13$
$\binom{6}{2} = 3.5$	$\binom{15}{2} = 3.5.7$
$\binom{7}{3} = 5.7$	$\binom{15}{6} = 5.7.11.13$
$\binom{10}{4} = 2.3.5.7$	$\binom{21}{2} = 2.3.5.7$

Table 5

Values of V(n,k), where they are consecutive integers.

k \ n	4	9	11	27	99	420	468	503
1	1	1	1	1	2	4	3	1
2	2	2	2	2	3	5	4	2
3		3	3	3	4	6	5	3
4			4	4	5	7	6	4
5				5	6	8	7	5
6				6	7	9	8	6
7					8	10	9	7
8					9		10	
9							11	
10							12	

APPENDIX

Values of k *for which* $n_k \le k^2$.

While we were searching for k's for which $n_k \le k^2$, by sheer brute force, Ernst S. Selmer, working on the UNIVAC 1110 at the University of Bergen, completed his project of computing n_k for $k \le 200$. His table shows that (within its limits)

$$n_k \le k^2 \text{ only for } k = 2,3,\dots,30,32,36,37.$$

It is almost certain that this list is complete. Our thanks are due to Selmer for his making a copy of his work available to us. His table also brought to light a small slip we had made in computing m_{30}.

The only note-worthy facts that our calculations have brought out are:

(i) $m_{51} = 3446 > n_{51} = 3445$;

(ii) $\binom{1007}{30}$ is square-free.

The relevant factorizations are:

$$\binom{3446}{51} = 1723.53.313.1721.181.191.491.859.229.101.3433.$$
$$73.7.127.857.149.571.137.107.163.59.311.263.1709.$$
$$67.61.683.569.3413.853.379.487.71.3407.131.227.$$
$$83.3^3.179.103.1699.79.283.2^2.11.13.19.31.37.$$
$$41.43;$$

$$\binom{3445}{51} = 53.313.1721.181.191.491.859.229.101.3433.73.7^2.$$
$$127.857.149.571.137.107.163.59.311.263.1709.$$
$$67.61.683.569.3413.853.379.487.71.3407.131.227.$$
$$83.3^3.179,103.1699.79.283.97.2.5.11.13.19.31.$$
$$37.41.43;$$

$$\binom{1007}{30} = 53.503.67.251.59.167.5.37.499.997.83.199.71.$$
$$331.2.31.991.43.47.197.41.983.491.109.7.89.$$
$$163.11.17.19.$$

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