# Some Extremal Problems in Geometry III

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## 1. Introduction

Let $X_1, X_2, \ldots, X_n$ be distinct points in k-dimensional Euclidean space $E_k$, let $d(X_i, X_j)$ denote the distance between $X_i$ and $X_j$, and let $g_k(n)$ denote the maximum number of solutions of $d(X_i, X_j) = a$, $1 \leq i < j \leq n$, where the maximum is taken over all possible choices of $a$ and distinct $X_1, \ldots, X_n$. In words, $g_k(n)$ is the maximum number of times that the same distance can occur among $n$ points in $E_k$. One of the authors proved in [1] that

$$g_2(n) > n^{1+c/\log\log n} .$$

(Throughout this report $c$ and $c_i$ denote positive constants not necessarily the same at every occurrence).

Szemerédi proved recently in [9] that $g_2(n) = o(n^{3/2})$, and one of the authors has shown in [2] that

$$c_1 n^{4/3} < g_3(n) < c_2 n^{5/3}$$

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and

$$\lim_{n\to\infty} g_k(n)/n^2 = (1/2) - \frac{1}{2[\frac{k}{2}]}$$

for $k \geq 4$, where $[x]$ denotes the integer part of $x$.

In other work [4], [7] the authors discuss the maximum number of times $f_k^a(n)$ that the same non-zero area can occur among the triangles $\Delta X_i X_j X_\ell$ $1 \leq i < j < \ell \leq n$, where the maximum is again taken over all choices for $X_1, \ldots, X_n$ in $E_k$.

In this report we discuss the maximum number $f_k^i(n)$ isosceles triangles that can occur (congruent or not), the maximum number $f_k^e(n)$ of equilateral triangles that can occur, the maximum number $f_k^c(n)$ of pairwise congruent triangles, and the maximum number $f_k^s(n)$ of pairwise similar triangles that can occur. All of these problems were posed at the end of our paper [4].

## 2. Isosceles Triangles

In the plane we have

**Theorem 1.**

$$c_1 n^2 \log n < f_2^i(n) < c_2 n^{5/2}$$

Proof 1. Let $X_0, X_1, \ldots, X_n$ be distinct points in $E_2$. For $1 \leq i \leq n$, the points forming an isosceles triangle with $X_0$ and $X_i$ on the base lie on a line, and these lines are distinct. Let $v_i$ denote the number of points $X_j$ on the ith line. The number of isosceles triangles having $X_0$ as a base vertex is $\sum_{i=1}^{n} v_i$, and it will be enough to show that this is less than $cn^{3/2}$. The lines containing fewer than $\sqrt{n}$ points clearly present no difficulty. Let $k \geq 0$ be fixed, and suppose that $v_{i_1}, \ldots, v_{i_N}$ are the $v_i$ satisfying $2^k\sqrt{n} \leq v_i < 2^{k+1}\sqrt{n}$, where $N = N_k$.

Since two lines have at most one point in common, we have

$$\sum_{j=1}^{N_k} \binom{v_{i_j}}{2} \leq \binom{n}{2} .$$

Using the inequalities on $v_{i_j}$,

$$N_k \frac{1}{2} 2^k\sqrt{n}(2^k\sqrt{n} - 1) \leq \binom{n}{2} ,$$

$$N_k < \frac{cn}{4^k} ,$$

$$\sum_{j=1}^{N_k} v_{i_j} \leq N_k 2^{k+1}\sqrt{n} < cn^{3/2}/2^k ,$$

and summing over $k$ gives the result.

2. Let $m = [\sqrt{n}]$ and consider the points $X_i = (u_i, v_i)$ with integer coordinates satisfying $|u_i|, |v_i| \leq m/2$. Let $u$ and $v$ be fixed, $|u|,|v| \leq m/4$. If $k < m^2/16$, then the circle with center $(u,v)$ and radius $\sqrt{k}$ will lie inside the region

$$R = \{(x,y): |x|, |y| \leq m/2\} ,$$

and the number of points $X_i$ lying on the circle will be $r(k)$, the number of representations of $k$ in the form $k = \ell^2 + m^2$, where $\ell$ and $m$ are integers. The pairs of points on the circle give us $\binom{r(k)}{2}$ isosceles triangles having $(u,v)$ as a vertex. Hence there are at least $\sum_{k=1}^{N} \binom{r(k)}{2}$ isosceles triangles having

$(u,v)$ as a vertex, where $N \geq [m/4]^2 > cn$. By formula 22 of [8] and (18.7.1) of [5], we have

$$\sum_{k=1}^{N} \binom{r(k)}{2} = \frac{1}{2}\sum_{k=1}^{N} r^2(k) - \frac{1}{2}\sum_{k=1}^{N} r(k)$$

$$= \frac{N}{8}(\log N + B) + O(n^{3/5 + \varepsilon})$$

$$- \frac{1}{2}\pi N + O(N^{1/2})$$

for every $\varepsilon > 0$, where $B$ is a constant. Hence the number of isosceles triangles containing $(u,v)$ is at least $cn \log n$. There are $cn$ choices for $(u,v)$ and the result follows.

Theorem 2. $f_3^i(n) \geq 2n^3/27 - cn^2$

Proof. Let $n$ be given, and let $X_i = (u_i, v_i, 0)$ for $1 \leq i \leq [2n/3]$, where $u_i, v_i$ are distinct solutions of $u^2 + v^2 = 1$, and let

$$Y_i = (0,0,i) \quad \text{for} \quad 1 \leq i \leq n - [2n/3] .$$

The triangles $\Delta X_i X_j Y_k$ for $1 \leq i < j \leq [2n/3]$ and $1 \leq k \leq n - [2n/3]$ are isosceles; hence

$$f_3^i(n) \geq \frac{1}{2}((2n/3) - 1)((2n/3) - 2)(n/3)$$

$$\geq (2/27)n^3 - cn^2 .$$

## 3. Equilateral Triangles

In the plane we have

**Theorem 3.** $\frac{1}{6} n^2 - cn^{3/2} \leq f_2^e(n) \leq n^2/3$

Proof 1. Let $X_1, X_2, \ldots, X_n$ be distinct points in $E_2$. For fixed $X_i$ and $X_j$ there are at most two points $X$ such that $\Delta X_i X_j X$ is equilateral. Hence $f_2^e(n) \leq \frac{2}{3}\binom{n}{2}$ , and the result follows.

2. Let $\Lambda$ be the geometrical lattice known as the triangular or 60° lattice. Let $n$ be given, and let $\rho$ be a positive number chosen so that the unit disc centered on the origin contains between $n - c_1\sqrt{n}$ and $n + c_2\sqrt{n}$ points of $\rho\Lambda$ . If $X$ and $Y$ are in $\rho\Lambda$, then both of the points $Z$ forming equilateral triangles with $X$ and $Y$ will lie in $\rho\Lambda$, but not necessarily in the unit disc.

It is convenient to think of the points as complex numbers. Let $z$ be a fixed point in the unit disc. If $w$ is also in the unit disc , the point

$$\xi = \frac{1}{2}(z + w) + i\frac{\sqrt{3}}{2}(z - w)$$

forms an equilateral triangle with $z$ and $w$. The requirement that $|\xi| \leq 1$ restricts

$$w = -\frac{(1 + i\sqrt{3})\xi}{2} - \frac{(1 + i\sqrt{3})}{2} z$$

to lie in a disc of radius one and center $\frac{(1 + i\sqrt{3})}{2} z$ .

The area in which this disc intersects the disc $|w| \leq 1$ is the area of overlap of two unit discs whose centers are distance $\left|\frac{(1 + i\sqrt{3})}{2} z\right| = |z|$ apart. If $z = x + iy$, this area is easily seen to be

$$A(x,y) = 2\int_0^{\sqrt{1-x^2-y^2}} \{2\sqrt{1-z^2} - \sqrt{x^2+y^2}\}\, dz .$$

If $z$ is a point of $\rho\Lambda$ having modulus less than one, then the number of equilateral triangles having $z = x + iy$ as a vertex is at least $\frac{A(x,y)\, n}{\pi} - c\sqrt{n}$ .

By integrating this function over the unit disc, and bearing in mind that every triangle is obtained three times in this way, we get $f_2^e(n) \geq \frac{n^2}{3\pi}2\ I - cn^{3/2}$ , where

$$I = \int_{x^2+y^2\leq 1} A(x,y)\,dxdy.$$

Hence

$$I = 2\int_{x^2+y^2\leq 1} dxdy\int_0^{\sqrt{1-x^2-y^2}} \{2\sqrt{1-z^2} - \sqrt{x^2+y^2}\}\ dz$$

$$= 4\pi\int_0^1 rdr\int_0^{\sqrt{1-r^2}} \{2\sqrt{1-z^2} - r\}\ dz$$

$$= 4\pi\int_0^1 r\ \mathrm{Sin}^{-1}(\sqrt{1-r^2})\ dr$$

$$= 4\pi\int_0^1 t\ \mathrm{Sin}^{-1}t\ dt$$

$$= [2\pi t^2\ \mathrm{Sin}^{-1}t]_0^1 - 2\pi\int_0^1 t^2dt/\sqrt{1-t^2}$$

$$= \pi^2 - \pi^2/2\ = \pi^2/2\ .$$

Hence

$$f_2^e(n) \geq (n^2/3\pi^2)(\pi^2/2) - cn^{3/2}$$

$$= (n^2/6) - cn^{3/2}$$

as claimed.

In space, we have

$$f_3^e(n) \leq f_3^s(n) \leq cn^{7/3} \quad .$$

The second inequality will be proved in Section 4.

In $E_4$, we have

<u>Theorem 4</u>. $\quad f_4^e(n) \leq cn^{8/3}$ .

Proof. Let $X_0, X_1, \ldots, X_n$ be distinct points in $E_4$, and let $G$ be the graph whose vertices are $X_1, \ldots, X_n$ and whose edges are those $\overline{X_iX_j}$ for which $\Delta X_0X_iX_j$ is an equilateral triangle. We shall show that $G$ cannot contain a Kuratowski subgraph $K_{3,3}$.

Suppose that $G$ contains a $K_{3,3}$. Then there are points $Y_1, Y_2, Y_3, Z_1, Z_2$, and $Z_3$ such that the nine triangles $\Delta X_0Y_iZ_j$ are equilateral. They clearly must be congruent; let a denote their common side length. Let $1 \leq i \leq 3$ be fixed. The points $Z_j$, being equidistant from $X_0$ and $Y_i$, lie on a hyperplane $\pi_i$, which is the perpendicular bisector of the line segment $\overline{X_0Y_i}$. If we let $X_0$ be the origin of coordinates and let $\underset{\sim}{y}_i$ be the position vector of the point $Y_i$, then the points $Z_j$ lie on an ordinary sphere $s_i$, contained in $\pi_i$, with center $(1/2)\underset{\sim}{y}_i$ and radius $(\sqrt{3}/2)a$. For distinct $i$ and $j$, the spheres $s_i$, having different centers and equal radii, will intersect in a circle $c_{ij}$ with center $(1/2)(\underset{\sim}{y}_i + \underset{\sim}{y}_j)$. The two circles $c_{12}$ and $c_{13}$ have different centers, and yet they have three points $Z_j$ in common. This is clearly impossible; hence $G$ does not contain a $K_{3,3}$.

By a theorem of Turán, Sös, and Koväri [6] the graph $G$ has fewer than $cn^{5/3}$ edges; hence any vertex belongs to at most $cn^{5/3}$ equilateral triangles, and the result follows.

<u>Remark</u> By slightly elaborating the above argument, the following can be proved: If $X_1, \ldots, X_n$ are distinct points in $E_4$ and $\Delta XYZ$ is an acute or obtuse triangle, then no vertex can belong to more than $cn^{5/3}$ triangles similar to $\Delta XYZ$. The following

example shows that the assertion is not true if $\Delta XYZ$ is a right triangle:

Let $P: (0, 0, 0, 0)$

$X_i : (x_i, y_i, 0, 0) \quad 1 \leq i \leq n$

$Y_j : (0, 0, x_j, y_j) \quad 1 \leq j \leq n$

where $x_i^2 + y_j^2 = 1$. Then the $n^2$ triangles $\Delta PX_iY_j$ are all isosceles right triangles (and in fact, congruent).

In $E_5$ we have only $f_5^e(n) \leq f_5^s(n) \leq cn^{26/9}$, and the second inequality will be proved in Section 4.

In $E_6$, the following construction, which also appeared in [2] and [4], gives $m^3$ congruent equilateral triangles from only $3m$ points: For $1 \leq i \leq m$

$X_i : (u_i, v_i, 0,0,0,0)$

$Y_i : (0,0,u_i,v_i,0,0)$

$Z_i : (0,0,0,0,u_i,v_i)$ ,

where $u_i^2 + v_i^2 = 1$. The triangles $\Delta X_i Y_j Z_k$ are equilateral triangles with side one, and consequently $f_6^s(n)$, $f_6^e(n)$ and $f_6^c(n)$ are all greater than $(n^3/27) - cn^2$.

## 4. Similar Triangles

In the plane, we have

Theorem 5. $f_2^s(n) \leq cn^2$ .

Proof. Similar to the proof of Theorem 3, part one.

In space, we have

Theorem 6. $f_3^s(n) \leq cn^{7/3}$ .

Proof. Let $X_1, X_2, \ldots, X_n$ be distinct points in $E_3$, and let $\Delta ABC$ be a triangle (non-degenerate, of course).

If $i$ and $j$ are fixed, $1 \leq i < j \leq n$, then the locus of points $Z$ such that the vertices $X_i$, $X_j$ and $Z$, taken in some order, form a triangle similar to $\Delta ABC$ consists of at most a constant number $c$ circles. Let $N$ be the number of these circles over all $i$ and $j$, and let $v_i$ be the number of points $X_j$ on the $i$th circle. We have

$$N \leq cn^2 \quad ,$$

and since a triple of points can only occur on one circle, we have

$$\sum_{i=1}^{N} \binom{v_i}{3} \le \binom{n}{3} .$$

The number of triangles similar to $\Delta ABC$ is $\frac{1}{3} \sum_{i=1}^{N} v_i$, and the maximum of this function, even allowing positive real $v_i$, subject to the constraint

$$\sum_{i=1}^{N} v_i(v_i - 1)(v_i - 2) \le 6\binom{n}{3}$$

occurs when the $v_i$ are all equal, because the function on the left-hand side is convex. Consequently,

$$\frac{1}{3} \sum_{i=1}^{N} v_i \le \frac{N}{3} \{2 + \{\frac{n(n-1)(n-2)}{N}\}^{1/3} \}$$

$$= \frac{2}{3}N + \frac{1}{3} \{n(n-1)(n-2)\}^{1/3} N^{2/3}$$

$$\le cn^{7/3}$$ , by the upper bound on $N$.

**Theorem 7.** $f_4^s(n) \leq cn^{17/6}$

**Proof.** Let $\Delta ABC$ be a non-degenerate triangle, and let $X_1, X_2, \ldots, X_n$ be in $E_4$ and distinct. We form the 3-graph $G$ whose vertices are the $X_i$, and whose edges are the unordered triples $\{X_i, X_j, X_k\}$ such that $\Delta X_i X_j X_k$ is similar to $\Delta ABC$. We claim there cannot be a $K_3(2,3,3)$ subgraph of $G$. That is, there cannot be vertices $Y_1, Y_2, Z_1, Z_2, Z_3, W_1, W_2$, and $W_3$ such that the 18 triples $\{Y_i, Z_j, Z_k\}$ for $1 \leq i \leq 2$, $1 \leq j$, $k \leq 3$ are all in $G$. Suppose that such $Y_i, Z_j, W_k$ do exist. Then the triangles $\Delta Y_i Z_j W_k$ are similar to $\Delta ABC$ and all congruent to each other. The three points $Z_j$ lie on a hypersphere, they are not collinear, and they determine a two-dimensional plane $\pi_z$. The three points $W_k$ determine, similarly, a two-dimensional plane $\pi_w$, and the two points $X_i$ determine a line $\ell$. Since the $Z_j$ are equidistant from the $Y_i$, $\pi_z$ must be orthogonal to $\ell$.

Similarly, $\pi_w$ is orthogonal to $\ell$ and $\pi_z$. This is only possible in five or more dimensions; hence the $K_3(2,3,3)$ does not occur, as claimed. It follows from the methods of [6] and [3] that $G$ has fewer than $cn^{3-\frac{1}{k\ell}}$ edges if $G$ contains no $K_3(k,\ell,m)$, where $c$ depends only on $k$, $\ell$ and $m$. Consequently, there are fewer than $cn^{17/6}$ triangles similar to $\Delta ABC$.

<u>Theorem 8</u>. $f_5^s(n) \leq cn^{26/9}$ .

<u>Proof.</u> Similar to the proof of theorem 7.

The 3-graph $G$ does not contain a $K_3(3,3,3)$ , and therefore $G$ has fewer than $cn^{26/9}$ edges.

## 5. Congruent Triangles

In the plane, we have

<u>Theorem 9</u>. $f_2^c(n) = o(n^{3/2})$ .

<u>Proof.</u> Let $\Delta ABC$ be an arbitrary non-degenerate triangle, and let $X_1, \ldots, X_n$ be distinct points in the plane. The result $g_2(n) = o(n^{3/2})$ , due to Szemerédi, which was mentioned in Section 1, implies that no more than $o(n^{3/2})$ pairs $\{X_i,X_j\}$ can be at distance $\overline{AB}$. Each pair can occur in at most $c$ triangles congruent to $\Delta ABC$ , and the result follows.

In space, we have

<u>Theorem 10</u>. $f_3^c(n) \leq cn^{19/9}$ .

Proof. Let $\Delta ABC$ be an arbitrary non-degenerate triangle, and let $X_1, \ldots, X_n$ be distinct points in space. The result $g_3(n) < c_2 n^{5/3}$ mentioned in Section 1 implies that no more than $cn^{5/3}$ pairs $\{X_i, X_j\}$ can be at distance $\overline{AB}$. For each such pair, the locus of points $X$ such that the vertices $X_i, X_j$ and $X$ taken in some order form a triangle congruent to $\Delta ABC$ consists of at most a constant number of circles. Let $N$ be the number of all of these circles as $\{X_i, X_j\}$ ranges over all the pairs at distance $\overline{AB}$. Then we have

$$N \leq cn^{5/3} \quad .$$

As in the proof of Theorem 6, we have $\sum_{i=1}^{N} \binom{v_i}{3} \leq \binom{n}{3}$ ,

where $v_i$ is the number of $X_j$ on the $i$th circle, and the number of triangles congruent to $\Delta ABC$ is at most

$$\sum_{i=1}^{N} v_i \leq 2N + \{n(n-1)(n-2)\}^{1/3} N^{2/3}$$
$$\leq cn^{19/9} \, .$$

## 6. Conclusion

In conclusion we would like to mention a few related problems. Throughout this section $\varepsilon$ will denote a positive number, not necessarily the same at every occurrence.

Is the inequality $f_6^e(n) \geq \frac{n^3}{27} - cn^2$ best possible? It would be interesting even to show $f_6^e(n) \leq (\frac{1}{6} - \varepsilon)n^3$.

What is the value of $\lim_{n\to\infty} f_2^e(n)/n^2$ ? Does the limit even exist? Can you prove $f_2^\varepsilon(n) \leq (\frac{1}{3} - \varepsilon)n^2$ ? Finally, we mention an entirely different problem: Given $n$ points in the plane, how many triangles $f_2(n)$ can approximate congruent equilateral triangles? By dividing the points into three small clusters we can get $f_2(n) \geq (n^3/27)$. It would be of interest to show $f_2(n) \leq (\frac{1}{4} - \varepsilon)n^3$.

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