COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
10. INFINITE AND FINITE SETS, KESZTHELY (HUNGARY), 1973.

PROBLEMS AND RESULTS ON FINITE AND INFINITE COMBINATORIAL ANALYSIS

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§0. INTRODUCTION

Hajnal and I have published two survey papers on problems in set theory* and I have published several surver papers on problems in graph theory and combinatorial analysis.** I cannot entirely hope to avoid repetitions but shall try to do so as much as possible and shall state old problems only if significant progress has been made on them, or if they seem to have been forgotten by everybody (sometimes including myself) and may deserve another attack.

*P. Erdős – A. Hajnal, Unsolved problems in set theory, *Proc. Symp. Pure Math.*, XIII part 1 Amer. Math. Soc., (1971) 17-48, and Solved and unsolved problems in set theory, *To appear in the Proceedings of the Tarski, Symposium*, held in Berkeley (1971).

**P. Erdős, Problems and results in combinatorical analysis, *Proc. Symp. Pure Math.*, XIX Amer. Math. Soc., 77-89, Some unsolved problems, *Michigan Math. J.*, 4 (1957), 291-300 and *Publ. Math. Inst. Hung. Acad. Sci.*, 6 (1961), 221-254; P. Erdős and D. Kleitman, Extremal problems among subsets of a set, *Combinatorial Math. and Applications, Proc. Chapel Hill Conference* 1970, 146-170; Some unsolved problems in graph theory and combinatorial analysis, *Combinatorial Math. and its Applications, Proc.* Conference Oxford, Editor Welsh, Acad. Press 1971, 97-109.

I certainly do not claim completeness but simply restrict myself to problems considered by my collaborators and myself. Nor do I claim that these are more important than others I neglect, but they have from the point of view of the reader the advantage that I often know more about them than the reader.

The concepts will be defined in the text, proofs will usually not be given and references will be given at the end of each chapter.

§1. Δ-SYSTEMS AND RELATED PROBLEMS

A family of sets $\{A_\alpha\}$ is said to form a strong Δ-system if the intersection of any two of the A_α's is the same set. It forms a weak Δ-system if the cardinal number of the intersection of any two A_α's is the same, e.g. (1,2), (1,3) (2,3) form a weak Δ-system but not a strong one.

Rado, Milner and I investigated the following question: Let there be given m sets $\{A_\beta\}$ of size n. It is true that there are p of them which form a strong (respectively weak) Δ system? We completely solved these questions if $m \geq \aleph_0$, but tantalizing questions remain if m is finite. Denote by $f_s(k, l)$ the smallest integer for which for any choice of $f_s(k, l)$ sets of size k there are l of them which form a strong Δ-system and $f_w(k, l)$ is the analogous function for weak Δ-systems. (Here k and l are finite).

Rado and I proved

(1) $$l^k < f_s(k, l) < k!l^k \ .$$

We conjectured (c_1, c are absolute constants)

(2) $$f_s(k, l) < c^k l^k$$

and in fact it seems very likely that

(3) $$\lim_{k \to \infty} f_s(k, l)^{1/k} = c_l < \infty \ .$$

I offer 1000 Swiss Francs (or three ounces of gold whichever is worth more) for a proof or disproof of

(4) $$f_s(k, 3) < c^k \ .$$

The lower and upper bounds in (1) have been improved by Abbott and Hanson and Sauer but at this moment a proof or disproof of (3) is nowhere in sight. One would guess that the proof of $f_w(k, 3)$ will be much simpler than (4) but so far we have not even proved

$$f_w(k, 3) < k!^{1-\epsilon}$$

for some $\epsilon > 0$ and $k > k_0(\epsilon)$.

Abbott and Hanson proved

$$f_s(3, 3) = 20, \quad f_w(3, 3) = 11, \quad f_w(4, 3) = 25.$$

Denote by $f_s^{(l)}(n)$ (resp. $f_w^{(l)}(n)$) the largest integer t for which there are t subsets of a set S of size n so that no l of them form a strong (respectively weak) Δ-system. It is not hard to prove that

$$(1 + c_l')^n < f_s^{(l)}(n) < (1 + c_l'')^n$$

but the value of $\lim (f_s^{(l)}(n)^{1/n})$ is not known (it is strictly between (1) and (2).

Abbott recently observed that $f_w^{(l)}(n)$ leads to non-trivial questions. He observed that it is not trivial to show that $f_w^{(3)}(n) > cn$ for every c if n is sufficiently large. Szemerédi showed by an ingenious construction that

$$f_w^{(3)}(n) > n^{c \log\log n}$$

but we do not know if $f_w^{(3)}(n)$ increases exponentially.

In our paper with Milner and Rado we observe that if $|A_k| = n$, $1 \leqslant k < c2^n\sqrt{n}$ and the family $\{A_k\}$ forms a weak Δ-system then it also forms a strong Δ-system. The proof is very simple. Recently Lovász and I observed that the same result holds for k if $k < (2 - c_1)^n$. We now posed the following question: Denote by $F(n)$ the largest k for which there are k sets of size n which form a weak Δ-system but not a strong one. Estimate or determine $F(n)$. If $n = p + 1$ where p is a power of a prime, then the finite geometry gives $p^2 + p + 1 = n^2 + n - 1$, n-tuples any two of which have exactly one common element, in other words $F(n) \geqslant n^2 - n + 1$, perhaps $F(n) \leqslant n^2 - n + 1$ for all n equality if and only if there is a finite geometry. Unfortunately we could not decide this pretty conjecture.*

*Added in proof: Recently M. Deza, Solution d'un probleme de Erdős – Lovász, *J. Comb. Theory*, 16 (1974), 166-167, settled the problem in the affirmative.

Before ending this paragraph I would like to mention a rather technical problem on weak Δ-systems. Milner, Rado and I prove in our paper that assuming G.C.H. there are $\aleph_{\alpha+1}$ sets of size $\aleph_\alpha$ no three of which form a weak Δ-system if and only if $\aleph_\alpha = |\alpha|$. Further if $\aleph_\alpha = |\alpha|$ our $\aleph_{\alpha+1}$ sets are all subsets of a set of size $\aleph_\alpha$.

Assume now $\aleph_\alpha = |\alpha|$ and assume first that $\aleph_\alpha$ is inaccessible and $|S| = \aleph_\alpha$. By transfinite induction it is not difficult to construct $\aleph_{\alpha+1}$ almost disjoint subsets A_β of S, $|A_\beta| = \aleph_\alpha$, $\beta < \omega_{\alpha+1}$ so that if $\beta_1 < \beta_2 < \beta_3$ then $|A_{\beta_1} \cap A_{\beta_3}| \neq |A_{\beta_2} \cap A_{\beta_3}|$ thus a fortiori no three sets of our family form a weak Δ-system.

Assume next $\aleph_\alpha = |\alpha|$ but $\aleph_\alpha$ is not inaccessible. I have not been able to construct $\aleph_{\alpha+1}$ sets A_β, $\beta < \omega_{\alpha+1}$ of size $\aleph_\alpha$, so that for every $\beta_1 < \beta_2 < \beta_3$

$$|A_{\beta_1} \cap A_{\beta_3}| \neq |A_{\beta_2} \cap A_{\beta_3}|$$

and I doubt if such a family exists.

[1] P. Erdős – R. Rado, Intersection theorems for systems of sets I and II, *J. London Math. Soc.*, 35 (1960), 85-90 and 44 (1969), 467-479. The paper with Milner and Rado on weak Δ-systems will appear in the *Australian Journal of Mathematics.*

§2. PROPERTY B

A family of sets $\{A_\alpha\}$, $|A_\alpha|^{\geq 2}$ is said to have property B if there is a set S which meets every A_α and contains none of them. This concept is due to Miller. It is often more convenient to use a different terminology. A family of sets $\bar{F} = \{A_\alpha\}$ has chromatic number r if $\bigcup_\alpha A_\alpha$ can be divided into r sets S_i $(i < r)$ (r finite or infinite) so that no A_α is contained in any of the S_i, but such a division is not possible into fewer than r sets. We write $\chi(F) = r$. F has property B iff $\chi(F) = 2$. A family F is called uniform if all A_α's have the same power, it is called simple if $|A_\alpha \cap A_\beta| \leq 1$. A family of infinite sets is called almost disjoint if $|A_\alpha \cap A_\beta| < \min(|A_\alpha|, |A_\beta|)$. Miller proved that there is an almost disjoint family F of infinite subsets of the integers with $\chi(F) = \aleph_0$, on the other hand every family satisfying

(1) $\qquad |A_\alpha| = \aleph_0, \quad |A_{\alpha_1} \cap A_{\alpha_2}| < k < \aleph_0 \quad$ for $\quad \alpha_1 \neq \alpha_2$

has property B (i.e. is two chromatic).

An old problem of Hajnal and myself states: Is it true that for every infinite cardinal m there is a family F of almost disjoint denumerable sets of chromatic number m? Perhaps in fact the family can be contained in a set of power m. This conjecture was proved for $m < \aleph_0$ assuming $2^{\aleph_0} = \aleph_1$ Baumgartner, Devlin, Galvin and Hajnal, but is open for $m > \aleph_\omega$.

In fact they proved that for an $n < \omega$ there is a family of countable almost disjoint sets, in a set of cardinality $\aleph_n$ such that every subset of cardinality $\aleph_1$ contains an element of this family.*

Let $|A_\alpha| = M \geq \aleph_0$. The family $\{A_\alpha\}$ is said to be strongly almost disjoint if there is an $n > m$ so that for every α_1 and α_2 $|A_{\alpha_1} \cap A_{\alpha_3}| <$ $< n < m$ Hajnal and I thought that perhaps every strongly almost

*Added in proof: This problem has been answered affirmatively, see G. Elekes – G.I. Hofmann, On the chromatic number of almost disjoint families of countable sets. *This volume.*

disjoint family has property B. We proved this assuming G.C.H. if $n = \aleph_0$, $m = \aleph_1$ and the family has size $\leqslant \aleph_\omega$ (i.e. there are at most $\aleph_\omega$ sets in our family.) Our proof hopelessly breaks down if the size of the family is $\aleph_\omega$.

Hajnal and I also proved assuming C.H. that there is a family of $\aleph_1$ countable almost disjoint sets which does not have property B and the union of any $\aleph_1$ sets of our family has size $\aleph_1$. On the other hand we could not settle the following closely related question: Is there a family of $\aleph_1$ countable almost disjoint sets which does not have property B and any countable set "almost contains" at most $\aleph_0$ sets of our family? (S_1 almost contains S_2 if all but a finite number of elements of S_2 are in S_1).

Shelah and I proved the following conjecture of Hechler: Every family of denumerable almost disjoint sets no two of which are disjoint has property B. On the other hand we showed that there is a family of almost disjoint denumerable sets which do not have property B and so that there are no three sets in the family which are pairwise disjoint.

We also asked the following questions: Let $\{A_\alpha\}$ be a family of denumerable sets, assume that no A_α is contained in the union of a finite number of other A'_α's and $A_{\alpha_1} \cap A_{\alpha_2} \neq 0$ for $\alpha_1 \neq \alpha_2$. Is it then true that the family has property B? Hajnal and Shelah (independently) gave an affirmative answer. The problem could be generalized to higher cardinals e.g. $\{A_\alpha\}$ is a family of sets of power $\aleph_1$, no A_α is contained in the union of $\aleph_0$ other A_α's and $A_{\alpha_1} \cap A_{\alpha_2} \neq \phi$ for $\alpha_1 \neq \alpha_2$.

It is then true that the family has property B? As far as I know these questions are open.

Further we asked: Let $\{A_\alpha\}$ be a family of denumerable sets, no two disjoint and no A_α is contained in the union of $k < \omega$ or fewer other A's. Is it then true that $\{A_\alpha\}$ has property B? I believe this is still open for $k > 1$ Lovász and Shelah disproved it for $k = 1$ i.e. they constructed a three chromatic Sperner* family of countable sets. As far as I

*In a Sperner family of sets no one contains the other.

know it is not known if there is a three chromatic family of countable sets no one of which almost contains any other.

There are many interesting finite problems connected with property B many of them already stated in our paper with Hajnal. Denote by $m(n)$ the smallest integer for which there is a uniform family of $m(n)$ sets A_k, of size n, $1 \leq k \leq m(n)$ which does not have property B. $m(2) = 3$, $m(3) = 7$, $m(4)$ is unknown. Toft just showed $m(4) \leq 23$. W. Schmidt and I proved

$$2^n \left(1 + \frac{2}{n}\right)^{-1} < m(n) < cn^2 2^n .$$

It would be nice to give an asymptotic formula for $m(n)$. Denote by $m^*(n)$ the smallest integer for which there is a simple uniform family of $m^*(n)$ sets of size n not having property B. Jewitt and Hales, Abbott, Hajnal and I proved that $m^*(n)$ is finite, in fact Hajnal and I showed $m^*(n) < 11^n$ and Lovász and I proved $\lim_{n=\infty} m^*(n)^{1/n} = 4$. Several further questions and results on this subject are stated by Lovász and myself in our paper in this volume.

Before completing this chapter I state one of the unsolved problems of Hechler which seemed very interesting to me. Let F be a family of almost disjoint infinite subsets of the integers with the following property: Let B be any infinite subset of the integers then either there is an A in F with $A \subset B$, or there are a finite number of sets in F so that B is contained in their union. Hechler shows that if $2^{\aleph_0} = \aleph_1$ then there is such a family F of power $2^{\aleph_0}$. Is it true that if we split the k-tuples of the integers into two classes, there is a set $A \in F$ all whose k-tuples are in the same class? This problem is unsolved even for $k = 2$.

[1] P. Erdős – A. Hajnal, On a property of families of sets, *Acta Math. Acad. Sci. Hung.* 12 (1961), 87-123.

[2] P. Erdős – S. Shelah, Separability properties of almost disjoint families of sets, *Isreal Journal of Math.*, 12 (1972), 207-214.

[3] S. Hechler, Classifying almost disjoint families with applications to BN-N, *Israel Math. Journal,* 10 (1971), 413-432.

§3. CHROMATIC NUMBER OF GRAPHS AND SET SYSTEMS

First I state some well-known results. The girth of a graph is the length length of its smallest circuit, C_l denotes a circuit of size l and $K(n;m)$ denotes the complete bipartite graph of n white and m black vertices. For every integer k there is a graph $\mathcal{G}$ with $\chi(\mathcal{G}) = \aleph_0$ and girth k. On the other hand every $\mathcal{G}$ with $\chi(\mathcal{G}) > \aleph_0$ contains a $K(n;\aleph_1)$ for every integer n. For every cardinal number m and integer k there is a graph of size m, $\gamma(\mathcal{G}) = m$ and the smallest odd circuit of $\mathcal{G}$ has size $\geqslant 2k+1$. On the other hand to every $\mathcal{G}$ with $\chi(\mathcal{G}) \geqslant \aleph_1$ there is an integer n so that $\mathcal{G}$ contains a C_l for every $l > n$. Finally assuming C.H. there is a $\mathcal{G}$ with $|\mathcal{G}| = \chi(\mathcal{G}) = \aleph_1$ which does not contain a $K(\aleph_0, \aleph_0)$ and also contains no triangle.

These results are contained in the following papers:

[1] P. Erdős – R. Rado, A construction of graphs without triangles having preassigned order and chormatic number, *The Journal of London Math. Soc.*, 35 (1960), 445-448.

[2] Erdős – A. Hajnal, Chromatic number of graphs and set system tems, *Acta Math. Acad. Sci. Hung.*, 17 (1966), 61-99.

[3] P. Erdős – A. Hajnal – S. Shelah, On some general properties of chromatic numbers, Colloqu. Math. Soc. János Bolyai 8. *Topics in Topology*, Keszthely (Hungary) (1972), 243-255.

[4] A. Hajnal, A negative partition relation, *Proceedings of the National Academy*, 68 (1971), 142-144.

The simplest question which Hajnal and I could not solve states: Is it true that if $\chi(\mathcal{G}) = \aleph_1$ then there is an n_0 and an edge e of $\mathcal{G}$ so that for every $n > n_0$ there is a C_n in $\mathcal{G}$ containing e. We also do not know if there is a $\mathcal{G}$ with $\chi(\mathcal{G}) = \aleph_1$ not containing C_5 and a $K(\aleph_0, \aleph_0)$. The really fundamental problem here is due to Taylor: Let

$\chi(\mathcal{G}) = \aleph_1$ be an arbitrary graph. Is it true that for every m there is a $\mathcal{G}'$ with $\chi(\mathcal{G}') = m$ so that every finite subgraph of $\mathcal{G}'$ is contained in $\mathcal{G}$?

Our old results imply that a finite graph $\mathcal{G}$ has to be a subgraph of a graph of chromatic number $> \aleph_0$ if and only if $\mathcal{G}$ is bipartite. Now in trying to settle the problem of Taylor one could try to characterize the families F of finite graphs so that every graph of chromatic number $> \aleph_0$ contains at least one (or infinitely many, or all but a finite number) of the graphs from F. In a triple paper with Hajnal and Shelah we state some plausible conjectures concerning these questions. Now complications and interesting problems arise if we insist that the cardinal number of the vertices of the graph of chromatic number $\aleph_1$ should be $\aleph_1$ respectively $\aleph_n$. (It is an open problem if we get anything new by saying that the power of the vertices is $\leqslant \aleph_\alpha$ for $\alpha > \omega$.

Hajnal, Szemerédi and I proved (unpublished) that to every cardinal number m and to every $\epsilon > 0$ there is a graph $\mathcal{G}$ with $\chi(\mathcal{G}) = m$ such that every finite subgraph $\mathcal{G}_n$ of $\mathcal{G}$ of n vertices has an independent set os size $\left(\frac{1}{2} - \epsilon\right)n$. We do not know if this remains true if we insist that $\mathcal{G}$ has m vertices.

Also we have the following problem which seems very interesting to me: Let $\mathcal{G}$ be an infinite graph. $f_{\mathcal{G}}(n)$ is the largest integer so that $\mathcal{G}$ has a subgraph $\mathcal{G}'$ of n vertices and $\chi(\mathcal{G}') = f_{\mathcal{G}}(n)$. A well-known theorem of de Bruijn and myself implies that if $\mathcal{G}$ has infinite chromatic number then $\lim_{n \to \infty} f\ (n) = \infty$.

Is it true that there is a function $h(n)$, $h(n) \to \infty$ so that for every $\mathcal{G}$ of chromatic number $\aleph_1$, $f_{\mathcal{G}}(n) > h(n)$ for all $n > n_0(\mathcal{G})$? We know that if such an $h(n)$ exists that $h(n) = o(\log_k n)$ for every k ($\log_k n$) denotes the k-fold iterated logarithm). Observe that such an $h(n)$ does not exist if we only assume that $\mathcal{G}$ has infinite chromatic number (since there is a graph of infinite chromatic number of arbitrarily large girth).

Finally I state a few disconnected problems on chromatic graphs. Is it true that every graph of chromatic number $\aleph_1$ has a subgraph which cannot be disconnected by the omission of a finite set of vertices? This is a problem of Hajnal and myself. We know that there is a graph of chromatic number $\aleph_1$ every subgraph of which can be disconnected by the omission of a countable set of vertices.

Another conjecture of Hajnal and myself states: Let $\mathcal{G}$ be an m-chromatic graph (m is an infinite cardinal). Then it has a subgraph which contains no triangle and which is also m-chromatic. If $m = \aleph_0$ we conjectured that it has a subgraph $\mathcal{G}$ of girth k and chromatic number $\aleph_0$ (for every k).

Both conjectures have a finite form. We only state it about triangles: Is there a function $f(n)$, $f(n) \to \infty$ as $n \to \infty$ so that every $\mathcal{G}$ of chromatic number n contains a subgraph $\mathcal{G}'$ which has no triangle and has chromatic number $\geqslant f(n)$.

A probabilistic proof of this last conjecture might be possible if we could prove the following statement: there is a function $g(n) \to \infty$ so that every graph of chromatic number $\geqslant n$ contains a subgraph $\mathcal{G}'$ so that all but $o(2^{e(\mathcal{G}')})$ subgraphs of $\mathcal{G}'$ have a chromatic number $> g(n)$. ($e(\mathcal{G})$ is the number of edges of $\mathcal{G}$). That is, $\mathcal{G}'$ is a graph of large chromatic number almost all subgraphs of which also have a large chromatic number – e.g. the complete graph has this property. Unfortunately I can not decide about the truth of this conjecture.

Let $\mathcal{G}$ be a graph. Denote by $n_1 < n_2 < \dots$ the integers for which $\mathcal{G}$ contains a C_{n_i}. Hajnal and I conjectured that if $\mathcal{G}$ has chromatic number $\geqslant \aleph_0$ then $\sum \frac{1}{n_i} = \infty$. In this form the conjecture seemed unattackable, but perhaps the following sharper conjecture will be much easier: There is a $g(k)$ tending to infinite with k so that for every $\mathcal{G}(n, [kn])$ we have $\sum' \frac{1}{n_i} > g(k)$ where $\mathcal{G}(n, l)$ is a graph of n vertices and l edges, perhaps $g(k) = c \log k$.

Gallai constructed a four-chromatic $\mathcal{G}_n$ the smallest odd circuit of which has $n^{1/2}$ edges. Gallai and I then conjectured that for every k and $n > n_0(k)$ there is a k-chromatic $\mathcal{G}_n$ the smallest odd circuit of which has more than $c_1 n^{\frac{1}{k-2}}$ edges, and on the other hand if every odd circuit of $\mathcal{G}_n$ has more than $c_2 n^{\frac{1}{k-2}}$ edges than $\chi(\mathcal{G}_n) < k$. I claimed that I proved this for $k = 4$ but I was unable recently to reconstruct my proof. Thus perhaps my proof was not correct.

Finally I would like to call attention especially to two older problems of Hajnal and myself which seem fundamental to us: Assume G.C.H. Does there exist a graph $\mathcal{G}$ of power and chromatic number $\aleph_2$ every subgraph of which of power $\aleph_1$ has chromatic number $\aleph_0$? Is there a graph with $\aleph_{\omega+1}$ vertices of chromatic number $\aleph_1$ so that every subgraph of power $\leqslant \aleph_\omega$ has chromatic number $\leqslant \aleph_0$. Clearly both questions can be stated for general cardinal numbers.

Another problem of Hajnal and myself states:

Assume G.C.H. Define the vertices of $\mathcal{G}(\aleph_3)$ as sequences of integers of length ω_2. Two vertices are joined if the two sequences agree only in $\leqslant \aleph_1$ coordinates. We prove that the chromatic number of this graph is $\geqslant \aleph_1$. We show that it is consistent that its chromatic number is $\geqslant \aleph_2$. We do not know if its chromatic number can be $\aleph_3$.

Galvin asked: Is it true that $\chi(\mathcal{G})$ has the Darboux property? i.e. if $\chi(\mathcal{G}) = m$ and $n < m$ then $\mathcal{G}$ has a spanned subgraph $\mathcal{G}'$ with $\chi(\mathcal{G}') = n$? Galvin showed that it is consistent that this conjecture is false, but perhaps it is true if G.C.H. is assumed. The conjecture may follow without G.C.H. if $\mathcal{G}'$ can be any subgraph of $\mathcal{G}$ (not necessarily a spanned subgraph).

The situation about the chromatic number of set systems is much less clear. Galvin, Hajnal and I are publishing a long paper about these questions. First consider r-graphs with $r = 3$. In marked contrast to $r = 2$ if T_0 is a given finite triple system we do not in general know whether an $\aleph_1$-chromatic triple system contains T_0 as a sub-system, though we have many special results in this direction. For the many unsolved problems I refer to our paper which appears in this volume and also to the recent paper of Hajnal, Rotschild and myself. Here I only state one of our problems which seems very striking to me: Let S be a triple system defined on a set of power $\aleph_1$ which does not admit an independent set of power $\aleph_1$. Is it then true that S contains every finite system of triples no two of which have two elements in common.*

Before closing this chapter I only mention that a triple system of chromatic number $> \aleph_0$ no two triples of which have an edge in common must have cardinal number $\geqslant \aleph_2$ and using C.H. Hajnal, Rotschild and I construct such a system. Galvin, Hajnal and I construct for every infinite cardinal m a triple system of chromatic number m so that every subsystem for which no triples have an edge in common is two-chromatic (has property B). On the other hand if $\mathscr{G}$ has chromatic number $m \geqslant \aleph_0$ and no two triples of S have a pair in common then our S perhaps has an s-circuitless subsystem of chromatic number m (a triple system is s-circuitless if for every $t \leqslant s$ and t triples contain at least $2t+1$ elements.)

[1] Erdős – A. Hajnal, On the chromatic number of graphs and set-systems, *Acta. Math. Acad. Sci. Hung.*, 17 (1966), 61-99, On chromatic numbers of infinite graphs, *Graph theory Symposium held in Tihany, Hungary,* (1966), Akadémiai Kiadó (Budapest), *Academic Press.*, (New York), 83-98.

[2] T. Gallai, Kritische Graphen I., *Publ. Math. Inst. Hung. Acad.*, 8 (1963), 165-192.

[3] F. Galvin, Chromatic numbers of subgraphs, *P.M.H.*, 4 (1973), 117-119.

*Added in proof: We disproved this. See the Erdős, Galvin, Hajnal paper *in this volume.*

[4] P. Erdős – A. Hajnal – B. Rotschild, On chromatic number of graphs and set-systems. *Proceedings of the Cambridge Summer School*, (1971).

[5] P. Erdős – F. Galvin – A. Hajnal, On set-systems having large chromatic number and not containing prescribed subsystems. *This volume.*

R. Rado in a forthcoming paper studies some related questions.

1) Elekes, Hajnal and myself considered the following problem:

Let n be a given cardinal and $|S| = m$ be sufficiently large. Is it true that if we divide the denumerable subsets of S into n classes, there always are three sets, A, B, C in the same class so that all the unions $A \cup B$, $A \cup C$, $B \cup C$ are in the same class as the sets? This is the simplest case of several questions, instead of three sets we can ask for more and the union we can replace by other Boolean operations.

2) Hajnal and I proved the following theorem: Color the edges of $K(\aleph_1)$ by two colors so that neither color contains a $K(\aleph_0, \aleph_1)$. Then for every countable $\mathcal{G}$ there is a subgraph of our $K(\aleph_1)$ isomorphic to $\mathcal{G}$ all whose edges are coloured by I and all edges of the complementary graph by II. The following problem seems very hard and is perhaps undecidable: Is it true that if we color the edges of $K(\aleph_2)$ by two colors so that neither color contains a $K(\aleph_1, \aleph_2)$ then to every $\mathcal{G}$ of power $\leqslant \aleph_1$ there is an isomorphic subgraph of our $K(\aleph_2)$ so that all edges of it are coloured I and all edges of the complementary graph are coloured II. Many further problems can be asked but I have to refer to our forthcoming paper with Hajnal. I just state one extremely attractive question which we raised at least 10 years ago: Color the edges of $K(\aleph_1)$ by three colors so that every complete subgraph of size $\aleph_1$ contains edges of all three colors. Is it then true that there is a triangle all whose edges have different colors? I give 100 dollars for a proof or disproof, also for a proof of undecidability. Here also several generalisations are possible, also various finite forms but we do not discuss them here.*

3) Hindman recently proved the following conjecture of Rotschild and Graham: Let $|S| = \aleph_0$ and divide the finite subsets of S into two classes. Then there are infinitely many disjoint sets A_i $(i < \omega)$ so that all finite unions belong to the same class. (added in proof:

*Added in proof: Assuming C.H. S. Shelah proved that the answer is no for the last problem and that it is consistent to have a negative answer for the first one.

Baumgartner recently found a simple proof)

The following two questions can now be asked:

a) Is there an infinte cardinal m so that if $|S| = m$ and we divide all subsets A (or all countable subsets) of S into two classes then there are disjoint sets $A_k \subset S$ for $k < \omega$ so that all finite or infinite unions belong to the same class. My opinion is that this conjecture is wrong.

b) Is the following statement true? To every infinite cardinal n there is an m so that if $|S| = m$ and we divide the subsets (countable subsets?) of S into two classes there are n sets so that all finite unions belong to the same class.

I expect b) to be correct. Both a) and b) could be generalized if we divide the subsets into p classes but problem 2 shows that even for finite n new difficulties arise if $p \geqslant \aleph_0$.

4) Problem of Dowker. Let S be a set, I is a proper ideal of subsets of S and consider all set mappings $T(x)$: $x \in S$, $T(x)$ is in I.

We define two properties P_1 and P_2 of I. Property P_1 implies that $T(x)$ can be chosen so that no two elements of S are independent, property P_2 implies that $T(x)$ can be chosen so that for every decomposition $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \phi$, and every $x \in S_1$, $y \in S_2$ either $y \in T(x)$ or $x \in T(x)$. Clearly P_1 implies P_2. The problem now is, does P_2 imply P_1?

5) The theory of set mappings has been extensively studied since Turán raised the problem on infinite independent sets – here I just refer to the large literature, I only state a problem of Hajnal and myself: Assume G.C.H. and let $|S| = \aleph_{k+1}$. Does there exist a set mapping $T(X)$ which maps the $(k + 2)$-tuples of S into elements of S so that there should be no independent set of size $\aleph_1$? We proved this for $k = 0$, Hajnal showed that it is consistent for $k = 1$. Nothing is known for $k > 1$. We proved that if $|S| \geqslant \aleph_{k+2}$ then there always is an independent set of size $\aleph_1$.

6) It easy to prove by transfinite induction that if $|S| = \aleph_0$ or $|S| = \aleph_1$ (C.H. is assumed) then there is a family $= \{A_\alpha\}$ of countable subsets of S so that every $B \subset S$, $|B| = \aleph_0$ is the union of two disjoint A_α's uniquely. Does this still hold if $|S| = \aleph_2$? or in fact for $|S| > \aleph_2$?

7) Hajnal and I considered the following questions: Make correspond to every edge of a $K(\aleph_0)$ a subset of measure $> \alpha$ of $(0, 1)$. Is there an infinite path so that the sets corresponding to the edges of the infinite path have a common point? Another more recent question of ours states as follows: To every edge of a $K(\aleph_0)$ make correspond a finite subset of the integers so that none of these sets contain any other. Is it true that there always is an infinite path so that the complement of the union of the sets corresponding to the edges of our infinite path is infinite? *

Let $|S| = \aleph_1$. To each triplet of S make correspond a subset of $(0, 1)$ of measure $> \alpha$. Prove that there always is a quadruplet so that the sets corresponding to the four triplets of our quadruplet have a non-empty intersection.

*Added in proof: proved that the answer is negative.

Now we discuss some finite problems

§5. EXTREMAL GRAPH PROBLEMS

Many papers have recently appeared on this subject. Here I do not try to give a systematic treatment but just mention some recent problems which perhaps have nor yet been stated elsewhere:

1. Problem of Sauer and myself. Denote by $f_r(n)$ the smallest integer so that every $\mathcal{G}(n; f_r(n))$ contains a subgraph which is regular and of valence (or degree) r. Trivially $f_2(n) = n$, but already the determination of $f_3(n)$ seems to present great difficulties. We do not even have an asymptotic formula for $\log f_3(n)$.

2. Brown, V.T. Sós and I denote by $f_r(n; k, l)$ the smallest integer for which every $\mathcal{G}^r(n, f_r(n; k, l))$ contains a $\mathcal{G}^r(k; l)$ as a subgraph. We conjectured

$$(1) \qquad f_3(n; k, k-3) = o(n^2)$$

for every $k \geqslant 4$. This is trivial for $k = 4$ and 5. The first difficult case was $k = 6$. Szemerédi proved (1) a few weeks ago for $k = 6$. His ingenious proof utilizes his fundamental lemma which he used in proving

$$r_k(n) = o(n)$$

where $r_k(n)$ is the largest integer l for which there are l integers not exceeding n which do not contain an arithmetic progression of k terms.

We showed $f_3(n: 6, 3) > cn^{3/2}$ and thought that $f_3(n; 6, 3) < n^{2-c}$, but Ruzsa showed

$$f_3(n; 6, 3) > cnr_3(n) > n^{2-\epsilon}$$

for every $\epsilon > 0$. He also observed

$$f_3(n; 7, 4) > cnr_4(n) .$$

Perhaps $f_3(n; k, k-3) > c_k n r_{k-3}(n)$. At this moment $f_3(n; 7, 4) = o(n^2)$ is still open.

3. Problem of Czipszer, Hajnal and myself: Let $\mathscr{G}$ be a graph whose vertices are the integers. Denote by $f(n)$ the number of edges (i, j), $1 \leqslant i < j \leqslant n$. We conjecture that if for every $n > n_0$

$$f(n) > \frac{n^2}{2}\left(\frac{1}{2} - \frac{1}{2k} + \epsilon\right)$$

then $\mathscr{G}$ contains an increasing path of length k. We proved this for $k = 2$ and $k = 3$, $k > 3$ is open. It is easy to see that the result fails if ϵ is omitted.

4. Is it true that to every $\epsilon > 0$ there is an $f(\epsilon)$ so that every $\mathscr{G}(n; [n^{1+\epsilon}])$ contains a non-planar subgraph of fewer than $f(\epsilon)$ edges?

5. Finally I state a few extremal problems on bipartite graphs. M. Simonovits and I proved that every $\mathscr{G}(n; [cn^{8/5}])$ contains a cube. Is the exponent best possible? We can not even prove that for every c and $n > n_0(c)$ there is a $\mathscr{G}(n; [cn^{3/2}])$ which does not contain a cube.

I proved that every $\mathscr{G}(n; [c_3 n^{3/2}]$ contains a $\mathscr{G}(7; 9)$ of the following structure: $z, x_1, x_2, x_3, y_1, y_2, y_3$ are the vertices, z is joined to x_1, x_2, x_3 and each y_i is joined to two $x - s$ (two different y's to different x's). Is it true that to every $k > 3$ there is a c_k so that every $\mathscr{G}(n; [c_k n^{3/2}])$ contains a graph having the vertices $z; x_1, \ldots, x_k$, $y_1, \ldots, y_{\binom{k}{2}}$; z is joined to all the $x - s$ and every y is joined to two $x - s$ (different y's to different x's)? I can not do this for $k \geqslant 4$.

The following generalization just occurred to me: Is it true that every $\mathscr{G}\left(n; c_{k,r} n^{2 - \frac{1}{r}}\right]$ contains a graph of $1 + k + \binom{k}{r}$ vertices $z; x_1, \ldots, x_k$; $y_1, \ldots, y_{\binom{k}{r}}$ where z is joined to $x_1, \ldots, x_k$ and each y is jointed to r x's; distinct y's to distinct r-tuples? The easiest case seems to be $k = 4$, $r = 3$ but I have not yet done this either.

Simonovits, V.T. Sós and I recently considered the following question which we could not answer: Is it true that there is a c so that every $\mathscr{G}(n; [cn^{3/2}])$ contains the following bipartite graph of 10 vertices:

The white vertices are x_1, x_2, a, x_3, x_4, the black ones y_1, y_2, b, y_3, y_4, a is joined to all black vertices **except** b and b to all white **vertices** except a, (x_1, y_1, x_2, y_2) and (x_3, y_3, x_4, y_4) form a c_4. It perhaps seems more likely that the answer is **negative**. I proved that the **result** is affirmative if we only, consider the graph spanned by x_1, x_2, a, y_1, y_2, b, i.e. a $K(3, 3)$ minus an edge.

II. MISCELLANEOUS PROBLEMS ON FINITE SETS

1. Faber, Lovász and I conjectured that if

$$|A_k| = n, \qquad 1 \leq k \leq n \qquad \text{and} \qquad |A_i \cap A_j| \leq 1, \ 1 \leq i < j \leq y$$

then we can color the elements of $\bigcup_{k=1}^{n} A_k$ by n colors so that every A_k contains elements of all colors. It is surprising that this simple conjecture seems to be rather difficult. It clearly fails if we have $n + 1$ sets. Lovász and Greenwell proved it if the number of sets is $\leq \frac{n+1}{2}$.

We arrived at our conjecture from the following conjecture of W. Taylor: One can color the lattice points of n-dimensional space

$$(x_1, \ldots, x_n), \qquad 1 \leq x_i \leq t, \qquad t \geq 2^k$$

by k colors so that every line containing k of these points gets all the k colors. He proves this for many special cases. The first unsolved case is $n = 3$, $k = 9$.

2. Problem of Lovász and myself. Let $\{A_k\}$, $1 \leq k \leq t_n$ be a family of sets of size n no two of which are disjoint. Assume that if $|U| = n - 1$ there always is a member A_k of our family so that $|A_k \cap U| = \phi$. Determine or estimate min t_n.

We proved $t_n < cn^{3/2} \log n$, probably $t_n < cn \log n$ holds. In fact it seems to us that a random choice of $cn \log n$ lines in a finite projective plane will be such that no $n - 1$ points will represent the lines, but we hane not been able to prove this. Also we can not prove say $t_n \geq 3n$ for $n > u_0$. The sharpest result is

$$t_n > 2\frac{2}{3}n - 3.$$

3. Problem of Kneser. Let $|S| = 2n + k$. Define a graph of $\binom{2n+k}{n}$ vertices a s follows: The vertices of our graph are the n-tuples of S. Two vertices are joined if the corresponding sets are disjoint. Prove that the chromatic number of this graph is $k + 2$. Clearly it is $\leqslant k + 2$. Szemerédi proved that the chromatic number is $\geqslant f(k)$ where $f(k)$ tends to infinity with k.

Let $A_i \subset S$, $1 \leqslant i \leqslant t$; $|A_i| = n$, $|A_i \cap A_j| \geqslant 1$ for $1 \leqslant i \leqslant j \leqslant t$. A well-known theorem of Ko, Rado and myself states that $t \leqslant \binom{2n+k-1}{n-1}$, equality if all the A_i have an element in common. Hajnal and I (and probably many others) considered the following more general question: Let

$$A_i^{(j)}, \quad 1 \leqslant i \leqslant t_j, \quad 1 \leqslant j \leqslant l \leqslant k$$

be distinct subsets of S, where $|S| = 2n + k$. Assume that for fixed j no two of the $A_i^{(i)}$ are disjoint. Determine $\max \sum_{j=1}^{l} t_j$. It would be nice if

$$(1) \qquad \max \sum_{j=1}^{l} t_j = \sum_{i=1}^{l} \binom{2n+k-i}{n-1}.$$

For $l = 1$ this is our theorem with Ko, and Rado. The general case would of course imply Knesers conjecture, but (1) has not been proved even for $l = 2$ and we have no real evidence for its truth. (added in proof) Hilton in fact shoved that (1) fails already for $l = 2$. Kneser's conjecture would follow from the weaker inequality

$$(2) \qquad \max \sum_{j=1}^{l} t_j < \sum_{i=1}^{l+1} \binom{n+k-i}{n-1}$$

and there is still some hope that (2) holds. Hajnal and Rothschild proved (1) for $n > n_0(k, l)$.