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DISTRIBUTION OF RATIONAL POINTS ON THE REAL LINE

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1. Introduction

Denote by $N_n(\alpha,\beta)$ the number of distinct fractions p/q, where $1 \leqq q \leqq n$ and $\alpha < p/q < \beta$. Let

$$D(\alpha) = \lim_{n\to\infty} \frac{1}{n} N_n\left(\alpha - \frac{1}{2n}, \alpha + \frac{1}{2n}\right).$$

It is shown in Sheng (1973) that

$$D(\alpha) = \frac{3}{\pi^2} \quad \text{if } \alpha \text{ is irrational}$$

and that

$$D\left(\frac{p}{q}\right) = \frac{2}{q} \sum_{r=1}^{[\frac{1}{2}q]} \left(1 - \frac{2r}{q}\right) \frac{\phi(r)}{r}$$

$$= \frac{3}{\pi^2} + O\left(\frac{\log q}{q}\right)$$

if $q > 1$ and $(p,q) = 1$. In this paper we prove two theorems.

THEOREM 1. *If* $(p,q) = 1$ *and* $q > 1$, *then*

$$\left| D\left(\frac{p}{q}\right) - \frac{3}{\pi^2} \right| < \frac{2}{q}\left(1 + \frac{2}{q}\right).$$

THEOREM 2. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences satisfying* $1 > \beta_n > \alpha_n > 0$ *and* $\lim_{n\to\infty} n(\beta_n - \alpha_n) = \infty$. *Then*

$$\lim_{n\to\infty} \frac{N_n(\alpha_n,\beta_n)}{n^2(\beta_n - \alpha_n)} = \frac{3}{\pi^2}.$$

In other words, the distribution of fractions is uniform over sufficiently long intervals.

Throughout this paper, $\mu(n)$ denotes the Möbius function, $\phi(n)$ denotes Euler's ϕ-function, and $[x]$ denotes the maximum integer $\leqq x$.

2. Lemmas

LEMMA 1. *Let n be a positive integer. Then*

$$\sum_{d=1}^{n} \mu(d)\left[\frac{n}{d}\right] = 1.$$

PROOF. This follows from

$$\mu(1) = 1$$

and, for $n > 1$,

$$\sum_{d=1}^{n} \mu(d)\left[\frac{n}{d}\right] - \sum_{d=1}^{n-1} \mu(d)\left[\frac{n-1}{d}\right] = \sum_{d|n} \mu(d) = 0.$$

LEMMA 2. *If $\lambda > 1$ and*

$$f(\lambda) = \sum_{r=1}^{[\lambda]} \left(1 - \frac{r}{\lambda}\right)\frac{\phi(r)}{r},$$

then

$$\left|f(\lambda) - \frac{3\lambda}{\pi^2}\right| < 1 + \frac{1}{\lambda}.$$

PROOF. Using $\phi(r) = r \sum_{d|r} \dfrac{\mu(d)}{d}$, we obtain (see Hardy and Wright (1960), page 268, lines 9–10)

$$\begin{aligned} f(\lambda) &= \sum_{d=1}^{[\lambda]} \mu(d)\left\{\frac{1}{d}\left[\frac{\lambda}{d}\right] - \frac{1}{2\lambda}\left[\frac{\lambda}{d}\right]^2 - \frac{1}{2\lambda}\left[\frac{\lambda}{d}\right]\right\} \\ &= \tfrac{1}{2}\lambda \sum_{d=1}^{[\lambda]} \frac{\mu(d)}{d^2} - \frac{1}{2\lambda} \sum_{d=1}^{[\lambda]} \mu(d)\left\{\frac{\lambda}{d} - \left[\frac{\lambda}{d}\right]\right\}^2 - \frac{1}{2\lambda} \sum_{d=1}^{[\lambda]} \mu(d)\left[\frac{\lambda}{d}\right]. \end{aligned}$$

By Lemma 1,

$$\begin{aligned} \left|f(\lambda) - \frac{3\lambda}{\pi^2}\right| &< \tfrac{1}{2}\lambda \sum_{d=[\lambda]+1}^{\infty} \frac{1}{d^2} + \frac{[\lambda]}{2\lambda} + \frac{1}{2\lambda} < \frac{\lambda}{2[\lambda]} + \frac{[\lambda]}{2\lambda} + \frac{1}{2\lambda} \\ &= 1 + \frac{(\lambda - [\lambda])^2}{2\lambda[\lambda]} + \frac{1}{2\lambda} < 1 + \frac{1}{\lambda}. \end{aligned}$$

LEMMA 3. *If $(p, q) = 1$ and $n \geqq qv > 0$, then*

$$N_n\left(\frac{p}{q}, \frac{p}{q} + \frac{v}{n}\right) = \frac{n}{q} \sum_{r=1}^{[vq]} \left(1 - \frac{r}{vq}\right)\frac{\phi(r)}{r} + O(vq \log vq). \tag{2.1}$$

PROOF. The proof is similar to that of Theorem 4 in Sheng (1973).

LEMMA 4. *If $(p, q) = 1$ and $n \geqq qv > 0$, then*

$$\frac{1}{n}N_n\left(\frac{p}{q},\frac{p}{q}+\frac{v}{n}\right)=\frac{3v}{\pi^2}+O\left(\frac{1}{q}\right)+O\left(\frac{vq\log vq}{n}\right). \tag{2.2}$$

PROOF. This follows from (2.1) and Lemma 2.

3. Proofs of theorems

PROOF OF THEOREM 1. This follows from

$$D\left(\frac{p}{q}\right)=\frac{2}{q}f\left(\frac{q}{2}\right)$$

and Lemma 2.

PROOF OF THEOREM 2. Given a positive integer n and real numbers α, β, γ satisfying

$$0<\alpha<\beta<1 \text{ and } \beta-\alpha=\frac{\gamma}{n}>\frac{1}{n},$$

we choose $\frac{p}{q}\in(\alpha,\beta)$ where

$$q\leqq y\,\forall\ \ \frac{x}{y}\in(\alpha,\beta),(x,y)=1,y\geqq 1.$$

Let $h/k<p/q<r/s$ be consecutive terms of the Farey sequence of order q. It is easy to see that

$$\frac{r}{s}-\frac{h}{k}=\frac{1}{sk}=\frac{v}{n}$$

for some real number v and that

$$\frac{h}{k}\leqq\alpha<\frac{p}{q}<\beta\leqq\frac{r}{s}.$$

Theorem 2 is proved if

$$\frac{1}{n\gamma}N_n(\alpha,\beta)=\frac{3}{\pi^2}+0\left(\frac{1}{\gamma}\right)+0\left(\frac{\log n}{n^{\frac{1}{3}}}\right) \tag{3.1}$$

holds.

We prove (3.1) in three possible cases.

CASE 1. Suppose $q\gamma\leqq n^{\frac{1}{3}}$. There exist $\xi\geqq 0$ and $\eta\geqq 0$ such that

$$\alpha=\frac{p}{q}-\frac{\xi}{n},\ \beta=\frac{p}{q}+\frac{\eta}{n},\ \xi+\eta=\gamma.$$

By Lemma 4,

$$\begin{aligned}\frac{1}{n}N_n(\alpha,\beta)&=\frac{1}{n}N_n\left(\frac{p}{q}-\frac{\xi}{n},\ \frac{p}{q}\right)+\frac{1}{n}N_n\left(\frac{p}{q},\ \frac{p}{q}+\frac{\eta}{n}\right)+\frac{1}{n}\\&=\frac{1}{n}N_n\left(\frac{q-p}{q},\ \frac{q-p}{q}+\frac{\xi}{n}\right)+\frac{1}{n}N_n\left(\frac{p}{q},\frac{p}{q}+\frac{\eta}{n}\right)+\frac{1}{n}\end{aligned}$$

$$= \frac{3}{\pi^2}(\xi + \eta) + O\left(\frac{1}{q}\right) + O\left(\frac{q\gamma \log q\gamma}{n}\right)$$

which can easily be reduced to (3.1).

CASE 2. Suppose $q\gamma > n^{\frac{1}{2}}$ and $k \leqq s$. Then there exist $\xi \geqq 0$ and $\eta > 0$ such that

$$\alpha = \frac{h}{k} + \frac{\xi}{n}, \beta = \frac{h}{k} + \frac{\eta}{n}, \eta - \xi = \gamma.$$

By Lemma 4,

$$\text{(3.2)} \qquad \frac{1}{n}N_n(\alpha, \beta) = \frac{1}{n}N_n\left(\frac{h}{k}, \frac{h}{n} + \frac{\eta}{n}\right) - \frac{1}{n}N_n\left(\frac{h}{n}, \frac{h}{k} + \frac{\xi}{n}\right)$$
$$= \frac{3}{\pi^2}(\eta - \xi) + O\left(\frac{1}{k}\right) + O\left(\frac{k\eta \log k\eta}{n}\right).$$

Clearly,

$$k\eta \leqq kv = \frac{n}{s} \leqq \frac{2n}{q} < 2n^{\frac{1}{2}}\gamma.$$

Thus

$$\frac{k\eta \log k\eta}{\gamma n} < \frac{2 \log (2n^{\frac{1}{2}}\gamma)}{n^{\frac{1}{2}}} = O\left(\frac{\log n}{n^{\frac{1}{2}}}\right).$$

It is now easy to deduce (3.1) from (3.2).

CASE 3. Suppose $q\gamma > n^{\frac{1}{2}}$ and $s < k$. Then there exist $\xi > 0$ and $\eta \geqq 0$ such that

$$\alpha = \frac{r}{s} - \frac{\xi}{n}, \beta = \frac{r}{s} - \frac{\eta}{n}, \xi - \eta = \gamma.$$

Here

$$\frac{1}{n}N_n(\alpha, \beta) = \frac{3}{\pi^2}(\xi - \eta) + O\left(\frac{1}{s}\right) + O\left(\frac{s\xi \log (s\xi)}{n}\right)$$

and (3.1) follows as in Case 2 from

$$s\xi < 2n^{\frac{1}{2}}\gamma.$$

This essentially proves Theorem 2.

One of us, T. K. Sheng, would like to take this opportunity to correct the following misprints in Sheng (1973): on page 244, the last term of (1.4) should read

$O\left(\frac{vq \log vq}{n}\right)$ instead of $O\left(\frac{vp \log vq}{n}\right)$; and on page 245, line 10 should read

$$D\left(\frac{p}{q}\right) = \frac{2}{q} \sum_{r=1}^{[\frac{1}{2}q]} \left(1 - \frac{2r}{q}\right) \frac{\phi(r)}{r}.$$