THE CHROMATIC INDEX
OF AN INFINITE COMPLETE HYPERGRAPH :
A PARTITION THEOREM

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1. Introduction. Let S be an infinite set of power n, and let $m \geqslant 2$ be an integer. We denote by $P_m(S)$ or $[S]^m$ the set of all subsets of S, of power m. A corollary of the main theorem proves that : if we denote by K_n^m the complete hypergraph having $P_m(S)$ as a set of edges (so its degree is n), then K_n^m has n for chromatic index. This result gives a positive answer to a conjecture of C. Berge.

2. Notations. Subsequently we assume the axiom of choice : particularly every infinite cardinal is an initial ordinal, denoted by ω_α. Moreover ω_o is written ω. If S is a set, its cardinality is denoted by $|S|$. If $m < |S|$ is a cardinal, then $P_m(S)$ or $[S]^m$ is the set of all subsets Y of S so that $|Y| = m$: an element of $[S]^m$ is called a m-tuple.

3. Theorem 1. Let p, m and n be three cardinals so that $p < m < n$ and $1 \leqslant p < \omega \leqslant n$. If S is a set of power n, there exists a partition $(\Delta_k)_{k \in I}$ of $[S]^m$, with $|I| = n^m$, such that for every k in I, every p-tuple is included in exactly one m-tuple, member of Δ_k.

If $p = 1$, each Δ_k defines a partition of S : the distinct sets, members of Δ_k, are disjoint, and Δ_k is a covering of S. This solves a conjecture of C. Berge : for $2 \leqslant m < \omega$ and $|S| \geqslant \omega$, the complete hypergraph has a coloring of the edges such that each vertex meets all the colors.

From $1 \leqslant p < \omega$, it follows that for any p-tuple Z of S and any m-tuple A of S, the condition $Z \subset A$ is equivalent to : $|Z \cap A| = |Z| = p$. So this result is a corollary of the following theorem :

4. Theorem 2. Let p , m and n be three cardinals so that $1 \leq p < m < n$ and $n^p = n \geq \omega$. If S is a set of power n , there exists a partition $(\Delta_k)_{k \in I}$ of $[S]^m$, with $|I| = n^m$, so that for every $k \in I$, on the one hand for every p-tuple Z of S , there is a m-tuple A , member of Δ_k , so that $|Z \cap A| = p = |Z|$, on the other hand, for distinct members A' and A'' of Δ_k , we have $|A' \cap A''| < p$.

In this theorem, and contrary to what happens in theorem 1, whenever $p \geq \omega$, we cannot suppose that every p-tuple is included in exactly one member of Δ_k . In fact, if we consider a p-tuple Z and suppose there is a unique set A in Δ_k which contains Z , then let z be an element of $S - A$, so $|Z \cup \{z\}| = p$, and there is a set A' , member of Δ_k , which contains $Z \cup \{z\}$. From $A' \neq A''$, we obtain a contradiction. Moreover, we cannot suppose that for every p-tuple Z in S , there is exactly one member A of Δ_k such that $|Z \cap A| = p = |Z|$: this is a consequence of the following remark : the union of two distinct p-tuples has p for power .

5. Proof of theorem 2. If such a partition exists, on one hand $n^p = n$, on the other hand for every distinct sets A' and A'' of Δ_k , we must have $|A' \cap A''| < p$. Therefore $|\Delta_k| = n$, and so $|I| = n^m$. Moreover $n^p = n = \omega_\eta$ and $n^m = \omega_\alpha$, so we denote by $(B_\xi)_{\xi < \omega_\alpha}$ an enumeration of $[S]^m$ and by $(Z_\lambda)_{\lambda < \omega_\eta}$ an enumeration of $[S]^p$. If m is finite, $m - p > o$ is an integer, otherwise $m - p$ is the cardinal m .

Suppose we defined the family $(\Delta_k)_{k < \gamma}$, when $\gamma < \omega_\alpha$, in such a way that we have :

a. for $k' < k'' < \gamma$, the sets $\Delta_{k'}$ and $\Delta_{k''}$ are disjoint.

b. for $k < \gamma$, if A' and A'' are distinct sets of Δ_k then $|A' \cap A''| < p$.

c. for $k < \gamma$, and for every p-tuple Z in S , there is at least one set A , member of Δ_k , such that $|Z \cap A| = p = |Z|$.

d. if $\xi < \gamma$, for some $k < \gamma$, the m-tuple B_ξ is a member of Δ_k .

We will construct the family $(\Delta_{\gamma,\rho})_{\rho < \omega_\eta}$ of sets of m-tuples such that the union of this family is Δ_γ .

1. If B_γ is a set, member of an already constructed Δ_k , then $\Delta_{\gamma,0}$ is the empty set.

2. If B_γ belongs to no Δ_k, for $k < \gamma$, then $\Delta_{\gamma,o}$ is the singleton $\{B_\gamma\}$.

Let λ' be the smallest λ so that $|Z_{\lambda'} \cap A| < p$ for every set A, member of $\Delta_{\gamma,o}$ (we have $\lambda' = o$ iff : $\Delta_{\gamma,o}$ is empty ; or $|Z_o \cap B_\gamma| < p$). If $\Delta_{\gamma,o}$ is the empty set, we put $S_o = Z_{\lambda'}$, otherwise we put $S_o = Z_{\lambda'} \cup B_\gamma$. For every $k < \gamma$ there is at most one subset C_k of S, member of Δ_k so that $Z_{\lambda'} \subset C_k$. We know that $[S - S_o]^{m-p}$ has $n^{m-p} = n^m$ elements and thus there is a subset D of $S - S_o$ verifying $|D| = m-p$ and so that : for every $k < \gamma$, the set $D \cup Z_{\lambda'}$ which is a m-tuple, is not a member of Δ_k. We remark that for every A, member of $\Delta_{\gamma,o}$, we have $|(Z_{\lambda'} \cup D) \cap A| = |Z_{\lambda'} \cap A| < p$. Let ξ' be the smallest ξ in ω_α so that :

1. for $k < \gamma$, the set $B_{\xi'}$ does not belong to Δ_k.
2. we have $|Z_{\lambda'} \cap B_{\xi'}| = p$.
3. for every A in $\Delta_{\gamma,o}$, we have $|B_{\xi'} \cap A| < p$ (this is verified whenever $\Delta_{\gamma,o}$ is the empty set).

We put $\Delta_{\gamma,1} = \Delta_{\gamma,o} \cup \{B_{\xi'}\}$

Suppose we defined $(\Delta_{\gamma,\nu})_{\nu < \rho}$, when $1 \leq \rho < \omega_n$, and verifying the following properties :

i. for $\nu' < \nu'' < \rho$ the set $\Delta_{\gamma,\nu'}$ is included in $\Delta_{\gamma,\nu''}$
ii. for distinct m-tuples A' and A'' in $\Delta_{\gamma,\nu}$, then $|A' \cap A''| < p$
iii. if A is a set of $\Delta_{\gamma,\nu}$, for every $k < \gamma$, the set A does not belong to Δ_k.

If ρ is a limit ordinal, then $\Delta_{\gamma,\rho}$ is the union, for $\nu < \rho$, of $\Delta_{\gamma,\nu}$.

If ρ is an isolated ordinal, $\rho = \theta + 1$, let S'_1 be the union of all members A of $\Delta_{\gamma,\theta}$. From $|A| = m < n$ and $|\rho| < n$, it follows that $|S'_1| < n$. Let λ'' be the smallest λ so that for every set A, member of $\Delta_{\gamma,\theta}$, we have $|Z_{\lambda''} \cap A| < p$. Such a λ'' exists because $|S - S'_1| = n$. We put $S_1 = Z_{\lambda''} \cup S'_1$. For every $k < \gamma$ there is at most a set D_k, member of Δ_k, so that $Z_{\lambda''} \subset D_k$. We know that $[S - S_1]^{m-p}$ has $n^{m-p} = n^m$ elements, and thus there is a (m-p)-tuple G of $S - S_1$ so that : on the one hand $|Z_{\lambda''} \cup G| = m$ (obvious), on the other hand $Z_{\lambda''} \cup G$ does not belong to every already constructed Δ_k. Moreover for every set A, member of $\Delta_{\gamma,\theta}$, we have $|Z_{\lambda''} \cup G) \cap A| = |Z_{\lambda''} \cap A| < p$. So let ξ'' be the smallest ξ so that :

i. for any $k < \gamma$, the set $B_{\xi''}$ is not a member of Δ_k.

ii. we have $|Z_{\lambda''} \cap B_{\xi''}| = p$.
iii. for every member A of $\Delta_{\gamma,\theta}$, we have $|A \cap B_{\xi''}| < p$.

Hence, we put $\Delta_{\gamma,\rho} = \Delta_{\gamma,\theta} \cup \{B_{\xi''}\}$.

From the construction, it follows that the family $(\Delta_{\gamma,\nu})_{\nu \leq \rho}$ verifies the conditions (i) , (ii) and (iii) . Moreover if the family $(\Delta_{\gamma,\rho})_{\rho < \omega_\eta}$ is constructed, then we put $\Delta_\gamma = \bigcup_{\rho<\omega_\eta} \Delta_{\gamma,\rho}$.

So the family $(\Delta_k)_{k \leq \gamma}$ verifies (a), (b), (c) and (d) .

Our transfinite induction is complete, and so the family $(\Delta_k)_{k<\omega_\alpha}$ verifies the conclusion of theorem 2 .

6. <u>The case when $n^p > n$</u> . Now, <u>we assume the general continuum hypothesis</u> (g.c.h.) . Let n and p be two infinite cardinals such that $n^p > n$. From g.c.h., it follows (by well known theorems [3]) that $n = \omega_\eta$ is a singular cardinal and that its cofinal type $cf(\omega_\eta) = \omega_\beta$ verifies $cf(n) = \omega_\beta \leq p = \omega_\delta$. Moreover, if $n = \omega_\eta$, then $n^+ = \omega_{\eta+1}$.

6.1 <u>Theorem 3.</u> <u>Let p , m and n be three infinite cardinals such that $\omega \leq p < m < n < n^p$, $cf(n) < p$ and $cf(n) \neq cf(p)$. If S is a set of power n , there exists a partition $(\Delta_k)_{k \in I}$ of $[S]^m$, with $|I| = n^m$, so that for every $k \in I$, on the one hand for every p-tuple Z of S , there is at least a m-tuple A , member of Δ_k , such that $|Z \cap A| = p = |Z|$ on the other hand, for distinct members A' and A" of Δ_k , we have $|A' \cap A''| < p$.</u>

<u>Proof</u>. Let S be a set of power n . So S is the union of an increasing family of sets $(S_\nu)_{\nu < \omega_\beta}$ such that : on one hand for $\nu' < \nu'' < \omega_\beta$ the set $S_{\nu'}$ is included in $S_{\nu''}$, on the other hand, for every $\nu < \omega_\beta$, we have $m < |S_\nu| = n_\nu < n$ and n_ν is a regular cardinal . From g.c.h., it follows that $[S_\nu]^p$ is a set of power n_ν . Let L be the union of $[S_\nu]^p$ for $\nu < \omega_\beta$, we have $|L| = n$. If we denote by $(Z'_\lambda)_{\lambda < \omega_n}$ an enumeration of L , by the method used in the proof of theorem 2 , we can construct a partition $(\Delta_k)_{k \in I}$ of $[S]^m$ such that for every k in I , on one hand, for every p-tuple Z' , member of L , there is a m-tuple A in Δ_k such that $|Z' \cap A| = p$, on the other hand, for distinct members A' and A" of Δ_k , we have

$$|A' \cap A''| < p$$

a. p is a regular cardinal. Let Z be a p-tuple in S , there exists a $\nu < \omega_\beta$ so that $Z \cap S_\nu = Z'$ verifies $|Z'| = p$: since $\omega_\beta = cf(n) < p$ and $p = cf(p)$. Therefore, there is at least a member A of Δ_k so that $|Z' \cap A| = p$ (indeed Z' belongs to $[S_\nu]^p$) and so $|Z \cap A| = p$. From these remarks , it follows that the family $(\Delta_k)_{k \in I}$ satisfies the conclusions of the theorem.

b. p is a singular cardinal such that $cf(n) < cf(p) < p$. Let Z be a p-tuple in S , there is some ν so that $|Z \cap S_\nu| = p$: otherwise let Z_ν be the set $Z \cap S_\nu$; from $|Z_\nu| < p$ and

$$Z = \bigcup_{\nu < \omega_\beta} Z_\nu$$

it follows that $|Z| < p$ (this is a consequence of $\omega_\beta = cf(n) < cf(p)$). So, we conclude as before .

c. p is a singular cardinal such that $cf(p) < cf(n) < p$. If Z is a p-tuple in S , there is at least a ν such that $|Z \cap S_\nu| = p$. Otherwise let Z_ν be the set $Z \cap S_\nu$, so Z is the union of Z_ν for $\nu < \omega_\beta = cf(n)$ and we have $|Z_\nu| = p_\nu < p$. It follows that p is the lower upper bound of the family $(p_\nu)_{\nu < \omega_\beta}$. Since ω_β is a regular cardinal, we can suppose that we have $p_{\nu'} < p_{\nu''}$ for $\nu' < \nu'' < \omega_\beta$. Therefore, $cf(n) = cf(p)$, and we have a contradiction . We conclude as before .

Remark. Under theorem hypotheses , if L is a subset of $[S]^p$ such that for every distinct members Z' and Z'' of L , we have $|Z' \cap Z''| < p$, then $|L| \leqslant n$. Otherwise $|L| \geqslant n^+ = 2^n$ and for every Z , member of L , let $\nu(Z)$ be a $\nu < \omega_\beta$ such that $|Z \cap S_\nu| = p$. From $n^+ = 2^n$ is a regular cardinal and from $cf(n) = \omega_\beta < n < n^+$, it follows that for some ν_o there are n^+ members Z of L such that $\nu(Z) = \nu_o$. Consequently there are at least two members Z' and Z'' of L such that $|S_{\nu_o} \cap Z' \cap Z''| = p$ (this is a consequence of $|[S_{\nu_o}]^p| = |S_{\nu_o}| < n^+$) , and we obtain a contradiction . This result is due to Tarski [4].

6.2 Theorem 4. Let p , m and n be three infinite cardinals such that $\omega \leqslant p < m < n < n^p$, and either $cf(n) = p$, or $cf(n) = cf(p)$. If S is a set of power n , there is no subset Δ of $[S]^m$ so that on one hand, for every p-tuple Z in S there is at least a member A of such that $|Z \cap A| = p$, on the other hand for distinct members A' and A'' of Δ , we have $|A' \cap A''| < p$.

Consequently, there is no partition $(\Delta_k)_{k\in I}$ of $[S]^m$ such that every Δ_k verifies the properties of the Δ above. Frascella, in [2], uses some similar idea.

Proof. First, we will prove that if such a Δ exists, then $|\Delta| \geq n^+ = 2^n$. To show this, we suppose $|\Delta| \leq n$, and thus $|\Delta| = n = \omega_\eta$. We denote by $(A_\xi)_{\xi<n}$ an enumeration of all members of Δ. We know that $n = \omega_\eta$ is the union of $cf(n) = \omega_\beta$ strictly increasing sets n_ν for $\nu < \omega_\beta$, with $|n_\nu| < n$.

a. if we have $cf(n) = p$, then we can construct, by transfinite induction, a sequence of elements w_ν in S, such that w_ν does not belong to the union V_ν of A_ξ for $\xi \in n_\nu$, and w_ν is distinct from every already constructed $w_{\nu'}$. This is possible : indeed let W_ν be the union of V_ν and the set of $w_{\nu'}$ for $\nu' < \nu$, we have $|W_\nu| < n$ and thus $|S - W_\nu| = n$. We denote by Z the set of all w_ν for $\nu < p$.

b. if we have $cf(n) = cf(p) < p$, then p is a singular cardinal. Let $(p_\nu)_{\nu<cf(p)}$ be a partition of p so that $|p_\nu| < p$ for $\nu < cf(p)$. We construct, by transfinite induction, a sequence of subsets Z_ν of S, for $\nu < cf(p)$, such that : on one hand Z_ν is disjoint from every already constructed $Z_{\nu'}$, on the other hand Z_ν is disjoint from the union of A_ξ for $\xi \in n_\nu$. We denote by Z the union of Z_ν for $\nu < cf(p)$.

In these two cases Z verifies $|Z| = p$. From the construction of Z, it follows that for every member A_ξ of Δ, we have $|Z \cap A_\xi| < p$. Contradiction. So $|\Delta| \geq n^+ = 2^n$.

Every set A_ξ, member of Δ, meets, for some $\nu < cf(n) = \omega_\beta$ the set S_ν in a set $B_{\xi,\nu}$ of power $\geq p$ (since $p < m$). From $n^+ = 2^n$, and from $\omega_\beta = cf(n) \leq p < n < n^+$, it follows that for some $\nu' < cf(n) < n^+$, there are n^+ sets A_ξ, members of Δ, such that $|B_{\xi,\nu'}| \geq p$. For such (ξ,ν') let $C_{\xi,\nu'}$ be a subset of $B_{\xi,\nu'}$ of power p. So $C_{\xi,\nu'}$ is included in $A_\xi \cap S_{\nu'}$ and $C_{\xi,\nu'}$ belongs to $[S_{\nu'}]^p$. Therefore, from $|S_{\nu'}| < n$, and so $|[S_{\nu'}]^p| < n < n^+$ (this is a consequence of g.c.h.) , it follows that there are two distinct sets $A_{\xi'}$ and $A_{\xi''}$, members of Δ, so that $C_{\xi',\nu'} = C_{\xi'',\nu'}$. So $|A_{\xi'} \cap A_{\xi''}| \geq p$, and we have a contradiction .

Remark. Under theorem hypotheses, if S is a set of power n, there is a subset L of $[S]^p$, of power $n^+ = 2^n$, such that for every distinct members Z' and Z" of L , we have $|Z' \cap Z''| < p$ (this result is in Tarski [4]) .

7. Problem. We don't know if the theorem 2 is true whenever $p = \omega$, $m = \omega_1$, $n = \omega_2$ and $n^p = \omega_3 = 2^p$: we do not suppose g.c.h. .

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