COMPLETE SUBGRAPHS OF CHROMATIC GRAPHS AND HYPERGRAPHS

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In this note we consider k-chromatic graphs with *given* colour classes $C_1,\ldots,C_k$, $|C_i| = n_i$. In other words, our k-chromatic r-graphs have vertex sets $\bigcup_1^k C_i$, and no r-tuple (i.e. edge of the r-graph) has two vertices in the same colour class C_i.

As usual, we denote by $K_p^{(r)}$ the complete r-graph with p vertices. The set of r-tuples of an r-graph G is denoted by E(G) and e(G) is the number of r-tuples. Let $\mathcal{G}$ be the set of k-chromatic r-graphs $\left(\textit{with } n = \sum_1^k n_i \textit{ vertices}\right)$ that do not contain a $K_p^{(r)}$. Let $m = \max\{e(G)\colon G \in \mathcal{G}\}$, $\mathcal{G}_{max} = \{G \in \mathcal{G}\colon e(G) = m\}$. We shall always suppose that $2 \le r < p \le k$.

The main aim of this note is to prove that $\mathcal{G}_{max}$ contains some special graphs. In particular, for $r = 2$ we obtain an extension of Turán's theorem [4]. We make use of some ideas in [1] and [3].

We call a k-chromatic r-graph G *canonical* if whenever $\left(v_{i_1},\ldots,v_{i_r}\right)$, $v_{i_j} \in C_{i_j}$, is an r-tuple of G then G contains *every* r-tuple of the form $\left(w_{i_1},\ldots,w_{i_r}\right)$, $w_{i_j} \in C_{i_j}$, $i = 1,\ldots,r$.

THEOREM 1. *$\mathcal{G}_{max}$ contains a canonical graph.*

Proof. Choose k algebraically independent numbers $\alpha_1,\ldots,\alpha_k$, $0 < \alpha_i < \varepsilon$, where $\varepsilon > 0$ satisfies $k^r n^r\{(1+\varepsilon)^r - 1\} < 1$. Define the weight of an r-tuple $\tau = \left(v_{i_1},\ldots,v_{i_r}\right)$ by

$$w(\tau) = \prod_{j=1}^{r}\left(1 + \alpha_{i_j}\right),$$ and the weight of a graph G by

$$w(G) = \sum_{\tau \in E(G)} w(\tau) .$$

Let $G_0 \in \mathcal{G}$ be a graph with maximal weight in $\mathcal{G}$. Note that $w(\tau) < (1+\varepsilon)^r$, so

$$e(G) \le w(G) < e(G) + 1$$

for every graph G. Thus $G_0 \in \mathcal{G}_{max}$.

We shall show that G_0 is canonical. Clearly it suffices to show that if $(x_1,\dots,x_r)$ is an r-tuple of G_0 and $\tilde{x}_1 \in C_1$ then $(\tilde{x}_1,x_2,\dots,x_r)$ is also an r-tuple of G_0. Suppose $(\tilde{x},x_2,\dots,x_r)$ is *not* an r-tuple of G_0. Then the algebraic independence of the α_i's implies that $d_w(x_1) \ne d_w(\tilde{x}_1)$, where

$$d_w(x) = \sum_{x \in \tau} w(\tau) .$$

We can suppose without loss of generality that $d_w(x_1) > d_w(\tilde{x}_1)$. Let $\tilde{G}_0$ be the graph obtained from G_0 by omitting the r-tuples containing $\tilde{x}_1$ and adding all the r-tuples of the form $(\tau-\{x_1\}) \cup \{\tilde{x}_1\}$, where $x_1 \in \tau$. Clearly $\tilde{G}_0 \in \mathcal{G}$ and $w(\tilde{G}_0) > w(G_0)$, contradicting the maximality of $w(G_0)$. Thus $G_0 \in \mathcal{G}_{max}$ is canonical, as claimed.

THEOREM 2. *Let* $r = 2$. *Then* $\mathcal{G}_{max}$ *contains a complete* (p-1)-*partite graph.*

Proof. Define G_0 as in the proof of Theorem 1. Then G_0 is canonical. Furthermore, if two vertices of G_0 are *not joined* then they must be joined to the *same* set of vertices, otherwise one could increase $w(G_0)$ by omitting the edges containing one of them and then joining this vertex to every vertex that is joined to the other one. This means exactly that G_0 is a complete s-partite graph for some s. As G_0 does not contain a $K_p \left(= K_p^{(2)}\right)$ one can take $s = p-1$.

If G is an edge-graph (i.e. 2-graph), denote by $\delta(G)$ the minimal degree of a vertex in G. Answering a question of Erdős, Graver showed (see [2]) that if G is a 3-chromatic edge graph, $n_1 = n_2 = n_3$ and $\delta(G) \geq n_1 + 1$ then G contains a triangle. A number of related problems were discussed by Bollobás, Erdős and Szemerédi [2].

Let us consider k-chromatic edge graphs with colour classes C_i, $|C_i| = n_i$, $i \in K = \{1,\dots,k\}$, $\sum_{i\in I} n_i = n$. Let

$$\xi = \left\{ I \subset K: \sum_{i\in I} n_i \leq n/2 \right\} \text{ and let } \delta_1 = \max\left\{ \sum_{i\in I} n_i : I \in \xi \right\}.$$

Let $K_1 \subset K$ be such that $\sum_{i\in K_1} = \delta_1$. Put $K_2 = K - K_1$. Joining every vertex of $\bigcup_{i\in K_1} C_i$ to every vertex of $\bigcup_{i\in K_2} C_i$ we obtain a graph $\tilde{G}$ with colour classes C_i, $i=1,\dots,k$. Clearly $\tilde{G}$ does not contain a triangle and $\delta(\tilde{G}) = \delta_1$. Extending the above mentioned result of Graver, we prove that this situation is best possible in a certain sense.

THEOREM 3. *Let* G *be a k-chromatic graph with colour classes* C_i, $|C_i| = n_i$, $i = 1,\dots,k$, *and* $\delta(G) \geq \delta_1+1$. *Then* G *contains a triangle.*

Proof. If $x \in \bigcup_{i\in K} C_i$ and $I \in \xi$, denote by $d_I(x)$ the number of vertices of $\bigcup_{i\in I} C_i$ joined to x. Let x_1 and $I_1 \in \xi$ be such that $d_{I_1}(x_1) \geq d_I(x)$ for every x and $I \in \xi$. Let $I_2 = K - I_1$. As $d_{I_1}(x_1) \leq \sum_{i\in I_1} n_i \leq \delta_1 < \delta_1+1$, there is a vertex $x_2 \in C_j$ that is joined to x_1 where $j \in I_2$. Then the maximality of $d_{I_1}(x_1)$ implies that $I_1 \cup \{j\} \notin \xi$, so $I_2' = I_2 - \{j\} \in \xi$.

Let us show that G has a triangle containing the vertices x_1, x_2. Suppose not. Then x_2 can be joined to at most $\delta_1 - d_{I_1}(x_1)$ vertices of $\bigcup_{i \in I_1} C_i$, so it must be joined to at least $\delta_1 + 1 - \left(\delta_1 - d_{I_1}(x_1)\right) = d_{I_1}(x_1) + 1$ vertices of $\bigcup_{i \in I_2'} C_i$. This contradicts the maximality of $d_{I_1}(x_1)$, so the proof is complete.

Remarks.

1. Theorem 2 can not be generalized for $r \geq 3$. It is false already in the simplest non-trivial case $r = 3$, $p = 4$, $k = 4$, $n_i = 1$. Then clearly a graph with 3 3-tuples does not contain a $K_4^{(3)}$. On the other hand, a 3-chromatic 3-graph with 4 vertices has at most 3-tuples.

2. The obvious generalization of Theorem 3 is also false. For $k \geq 4$ and sufficiently large m there is a k-chromatic edge graph G with m vertices in each colour class, such that $\delta(G) \geq (k-2)m + 1$ and G does not contain a K_k (see [2]).

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