A REMARK ON POLYNOMIALS AND THE TRANSFINITE DIAMETER

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ABSTRACT

The main result is the following theorem: Let E be the point set for which $\left| \prod_{\nu=1}^{n} (z-z_\nu) \right| < 1$. If the zeros z_ν $(\nu=1,\ldots,n)$ belong to a bounded, closed and connected set whose transfinite diameter is $1-c$ $(0<c<1)$, then E contains a disk of positive radius ρ, dependent only on c.

Let

$$f(z) = \prod_{\nu=1}^{n} (z - z_\nu), \tag{1}$$

and denote by $E = E(f)$ the point set for which

$$\left| f(z) \right| < 1. \tag{2}$$

The present note deals with the proof of the following theorem.

THEOREM. *Let D be a bounded, closed and connected set, whose transfinite diameter $d(D)$ is equal to $1-c$, $0<c<1$. Let $E(f)$ be the point set defined by* (2), *with $z_\nu \in D$, $\nu = 1,\cdots,n$. Then there exists a positive number $\rho = \rho(c)$ (dependent only on c) such that the set $E(f)$ always contains a disk of radius $\rho(c)$.*

A weaker result is proved in [1, Th. 6]. It should be added that we give here an existence proof. A numerical estimate for ρ and for the degree of the polynomial (mentioned below) would be interesting.

For the proof of the theorem we need the following:

LEMMA. *Let D be a set with the properties mentioned above. Then there*

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always exists a polynomial $P(z) = z^m + a_1 z^{m-1} + \cdots + a_m$, *whose degree* $m = m(c)$ *depends only on* c, *such that* $|P(z)| < \frac{1}{2}$ *on* D *(instead of* $\frac{1}{2}$ *we could take any fixed* a, $0 < a < 1$).

PROOF. Suppose to the contrary that the lemma were false. Then there would exist bounded, closed and connected sets $D_1, D_2, \cdots, D_n, \cdots$, all of them containing $z = 0$ with $d(D_n) = 1 - c$, $n = 1, 2, \cdots$, and such that any polynomial $P(z) = z^k + \ldots$ for which $\max_{z \in D_n} |P(z)| \leqq \frac{1}{2}$ is satisfied, must be of degree at least n.

Denote by F_n the component of the complement of D_n which contains $z = \infty$. Let the complement of F_n be D'_n. Evidently $D_n \subseteq D'_n$, and if $\max_{z \in D'_n} |P(z)| < \frac{1}{2}$, then the degree of P is $\geqq n$. By a well known theorem of Fekete [2], the univalent function $\zeta = f_n(z)$, which is regular in F_n except for $z = \infty$, where it has a simple pole with $f'_n(\infty) = 1$, maps F_n on $|\zeta| > 1 - c$. The inverse functions $z = \phi_n(\zeta)$ of $\zeta = f_n(z)$, $n = 1, 2, \cdots$, form a normal and compact family in $|\zeta| > 1 - c$. Hence a subsequence $\phi_{n_k}(\zeta)$, $k = 1, 2, \cdots$ converges uniformly in $|\zeta| > 1 - c + \varepsilon$ ($\varepsilon > 0$ and arbitrarily small so that $1 - c + \varepsilon < 1$) to a univalent function $\phi(\zeta) = \zeta + a_0 + a_1/\zeta + \cdots$ which maps $|\zeta| > 1 - c + \varepsilon$ on a domain F whose complement is D^* with $d(D^*) = 1 - c + \varepsilon$. The image of $|\zeta| = 1 - c + \varepsilon$ by $\phi(\zeta)$ is C which is the boundary of D^*. The analytic curve C_{n_k}, $k = 1, 2, \cdots$, which is the image of $|\zeta| = 1 - c + \varepsilon$ by $\phi_{n_k}(\zeta)$ is the boundary of a domain D^*_{nk} which contains D_{n_k}, $k = 1, 2, \cdots$. These domains D^*_{nk} converge uniformly to D^*; hence, because of our assumptions, there is no polynomial $P(z) = z^m + \cdots$ such that $|P(z)| \leqq \frac{1}{2}$ on D^*. But this last result, because of $d(D^*) = 1 - c + \varepsilon < 1$, is in contradiction to [2; §2, 3] and the proof of the lemma is complete.

The proof of the theorem follows now on the same lines as [1, Th. 6]. For the sake of completeness we present it here.

Let $P(z) = \Pi_{i=1}^{m} (z - t_i)$ be the polynomial of degree $m = m(c)$ whose existence was proved and which satisfies $|P(z)| < \frac{1}{2}$ on D. Evidently there exists a number $\rho > 0$ such that $\Pi_{i=1}^{m} |z - s_i| < \frac{1}{2} + \varepsilon$ for all z in D if s_i lies in the disk H_i whose radius is ρ and center is at t_i. Let $\max_{z \in H} |f(z)| = |f(s_i)|$.

Since

$$\prod_{i=1}^{m} f(s_i) = (-1)^{mn} \prod_{\nu=1}^{n} (z_\nu - s_1)(z_\nu - s_2) \cdots (z_\nu - s_m)$$

and since the right member is of a modulus less than 1, at least one of the quantities $|f(s_i)|$ is at most 1. Hence $|f(z)| < 1$ throughout one of the disks H_i, as was to be proved.

We remark that the theorem is false when D is not connected.

Indeed, consider the lemmiscate

$$|z^2 - a^2| < 1, \quad (a > 0).$$

By increasing a, it is seen that the radius of any disk contained in $E(z^2 - a^2)$ can be made as small as we please.

Our result implies, of course, that if D is a connected set of transfinite diameter $1 - c$ and if $z_\nu \in D$, then the area of the set $\overline{E(f)}$, given by $|\Pi^n_{\nu=1}(z - z_\nu)| \leqq 1$, is greater than $f(c)$; we have no explicit estimation of $f(c)$.

If D has transfinite diameter 1, then perhaps the area of $\overline{E(f)}$ can be made $< \varepsilon$ for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$ (here the connectedness of D will not be needed). That this is so when D is the unit circle or the interval $(-2, +2)$ is proved in [1]; the general case is open.

Another related problem is the maximum number of components of $\overline{E(f)}$. If D is the unit circle, it is proved in [1; Th. 7] that the maximum number is $n-1$, and if D is the interval $(-2, +2)$, it is easy to see that $E(f)$ can have n components. As far as we know, the general case has not been investigated.

References

1. P. Erdös, Herzog and G. Piranian, *Metric properties of polynomials*, J. Analyse Math. **6** (1958), 125–148.

2. M. Fekete, *Ueber die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. **17** (1923), 228–249.

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