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Two combinatorial problems in group theory

by

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Abstract. Sequences of elements from (additive) abelian groups are studied. Conditions under which a nonempty subsequence has sum equal to the group identity 0 are established. For example, an n-sequence with exactly k distinct terms represents 0 if the group has order $g \leqslant n + \binom{k}{2}$ and $n \geqslant k\binom{k}{2}$.

The least number $f(k)$ of distinct partial sums is also considered, for the case of k-sequences of distinct elements such that no nonempty partial sum is equal to 0. For example, $2k-1 \leqslant f(k) \leqslant [\frac{1}{2}k^2]+1$.

In this paper a *sequence* is a selection of members of a set, possibly with repetitions, in which order is not important; *elements* are members of sets, and *terms* are members of sequences.

DEFINITION. Let $*$ be a binary operation on a set A, and let $S = (a_i)_{i=1}^n$ be a sequence of elements from A. S will be said to *represent* the element $x \in A$ if

(i) x is a term in S, or

(ii) there exist $y, z \in A$ such that $x = y*z$, and y and z are represented by disjoint subsequences of S.

(Clearly this notion extends to general algebras.)

In particular, if $\langle G, +\rangle$ is an abelian group and $S = (a_i)_{i=1}^n$ is a sequence of elements from G, then S represents $x \in G$ just if there exists a sequence $E = (\varepsilon_i)_{i=1}^n$ of elements from $\{0, 1\}$, not all 0, such that $\sum_{i=1}^{n} \varepsilon_i a_i = x$.

We resolve here some aspects of the following two related problems.

(1) Under what circumstances does an n-sequence of elements from an abelian group represent the zero element?

(2) If an n-sequence of distinct elements from an abelian group does not represent the zero element, how many elements does it represent?

Sequences representing zero.

THEOREM 1. *Any n-sequence $S = (a_i)_{i=1}^n$ of elements from an abelian group $\langle G, +\rangle$, exactly k of which are distinct, represents the group identity 0 if the group has order $g \leqslant n + \binom{k}{2}$ and $n \geqslant k\binom{k}{2}$.*

Proof. Suppose on the contrary that S does not represent 0. Then, none of the elements represented by the first m terms of S is 0, and none is equal to any of the $n-m$ sums of the form $\sum_{i=1}^{r} a_i$, with $m+1 \leqslant r \leqslant n$, for otherwise the difference would be a sum equal to 0. Again, none of these latter $n-m$ sums equals 0, and all are distinct, for otherwise there would be a difference equal to 0, contrary to hypothesis.

We shall show that a suitable choice of m can be made, such that at least $m + \binom{k}{2}$ elements are represented by the first m terms of S, so with the latter $n-m$ sums a total of at least $n + \binom{k}{2}$ nonzero elements are represented. This is inconsistent with $g \leqslant n + \binom{k}{2}$, so the initial hypothesis is false and the theorem follows.

We may suppose there are t equal terms in S, say $a_i = a_1$ for $1 \leqslant i \leqslant t$, where $kt \geqslant n$. Then S represents those elements equal to sa_1, for $1 \leqslant s \leqslant t$, which are necessarily distinct and different from 0. There are now two cases to consider: either (i) S has a term not in $[a_1]$, the subgroup of G generated by a_1, or (ii) all terms of S are in $[a_1]$.

Case (i). Suppose $a_{t+1} \notin [a_1]$. Then with $m = t+1$, these first m terms of S must represent $2t+1$ distinct elements. If $n \geqslant k\binom{k}{2}$, at least $m + \binom{k}{2}$ distinct elements are represented, because $kt \geqslant n$, which is what we require.

Case (ii). Let $a_i = r_i a_1$ for $1 \leqslant i \leqslant n$, where the sequence $S' = (r_i)_{i=1}^n$ comprises positive integers, exactly k of which are distinct, and $r_i = 1$ for $1 \leqslant i \leqslant t$. (Since S does not represent 0, S' has no zero terms.) Regard S' as a sequence from the additive group of integers. If no term of S' exceeds t, then S' represents all positive integers up to and including $\sum_{i=1}^{n} r_i$, and this sum is at least as large as the sum of the first k positive integers together with a further $n-k$ ones. Thus S' certainly represents g if $g \leqslant n + \binom{k}{2}$, so S represents ga_1, which is 0 since g is a multiple of the order of a_1. This is a contradiction, so S' must contain a term which exceeds t, say $r_{t+1} > t$. Again take $m = t+1$ and repeat the argument of Case (i). ■

This theorem is best possible in the sense that the bound on g cannot be improved in general, for if $a_i = i$ for $1 \leqslant i \leqslant k$ and $a_i = 1$ for $k+1$

$\leqslant i \leqslant n$, then S represents all nonzero elements of the additive group of residue classes modulo $n+\binom{k}{2}+1$, but does not represent 0. On the other hand, it is not clear what the best bound for n should be. If we take G to be the additive group of residue classes modulo $2s^2+4s$, where s is any positive integer, and S to comprise $n = 3s$ terms specified by $a_i = i$ for $1 \leqslant i \leqslant s$, $a_i = 1$ for $s+1 \leqslant i \leqslant 2s-1$, and $a_i = s^2+i$ for $2s \leqslant i \leqslant 3s$, then $k = 2s+1$, and S does not represent 0. Since $g = 2s^2+4s = n+\binom{k}{2}$ in this case, it follows that the bound on n in Theorem 1 could not be reduced as far as $\frac{3}{2}(k-1)$ in general. We conjecture that the theorem is true for $n \geqslant ck$, where c is some positive constant.

It is desirable to obtain a result corresponding to Theorem 1 for the case in which the exact number of distinct elements appearing in S is not known, the only relevant information being a lower bound on the number. We can deduce this result by using a theorem first conjectured by Erdös and Heilbronn [2], and recently proved by Szemerédi [5], viz.,

THEOREM (Szemerédi). *Any k-sequence $S = (a_i)_{i=1}^{k}$ of elements from an abelian group $\langle G, +\rangle$, all of which are distinct, represents the group identity* 0 *if the group has order $g \geqslant g_0$ and $k \geqslant c_0\sqrt{g}$, where g_0 and c_0 are absolute constants.*

Thus, if the n-sequence S in Theorem 1 contains $h \geqslant k$ distinct elements, and the order of G satisfies $g_0 \leqslant g \leqslant n+\binom{k}{2}$, then the supposition that S does not represent 0 implies $h \leqslant c_0\sqrt{n+\binom{k}{2}}$. If t is the number of terms of S equal to a_1, we may assume $ht \geqslant n$, whence the argument used in Case (i) of the proof of Theorem 1 shows that the first m terms of S represent at least $m+\binom{k}{2}$ distinct elements provided $n \geqslant c_1 k^4$, where c_1 is an absolute constant. All other details of the proof carry over, so we have the

COROLLARY TO THEOREM 1. *Any n-sequence $S = (a_i)_{i=1}^{n}$ of elements from an abelian group $\langle G, +\rangle$, at least k of which are distinct, represents the group identity* 0 *if the order of the group satisfies $g_0 \leqslant g \leqslant n+\binom{k}{2}$ and $n \geqslant c_1 k^4$, where g_0 and c_1 are absolute constants.*

It is possible to obtain similar results even when the number of distinct elements in S is not so small in comparison with n.

THEOREM 2. *Any n-sequence $S = (a_i)_{i=1}^{n}$ of elements from an abelian group $\langle G, +\rangle$, at least k of which are distinct, represents the group identity* 0 *if the group has order $g \leqslant n+k-1$.*

Proof. Suppose on the contrary that S does not represent 0. We may take the first k terms of S to be distinct. As will be shown in the second part of this paper these k terms represent at least $f(k)$ distinct elements, and for any $k, f(k) \geqslant 2k-1$. None of the elements they represent can be equal to any of the $n-k$ sums $\sum_{i=1}^{r} a_i$, where $k+1 \leqslant r \leqslant n$, for otherwise the corresponding difference equals 0 and S would represent 0. Similarly, no two of the latter $n-k$ sums can be equal, so S represents at least $n+k-1$ distinct elements. Since S does not represent 0, this constitutes a contradiction if $g \leqslant n+k-1$, so the theorem follows. Indeed, it holds if $g \leqslant n+f(k)-k$. ∎

In a sense, the upper bound on g in Theorem 2 is low because of the structure of cyclic groups. This is clarified by the next result.

THEOREM 3. *Any n-sequence $S = (a_i)_{i=1}^{n}$ of elements from a noncyclic abelian group $\langle G, +\rangle$ represents the group identity* 0 *if the group has order $g \leqslant 2n-1$.*

This may readily be deduced from the following result of Olson [4]:

THEOREM (Olson). *If H, K are abelian groups of order h, k respectively, and $k \mid h$, then any n-sequence $S = (a_i)_{i=1}^{n}$ of elements from their direct sum $G = H \oplus K$ represents the identity $0 \in G$ if $n \geqslant h+k-1$.*

Proof of Theorem 3. If G is a noncyclic abelian group of finite order, there is a direct sum decomposition

$$G \cong \bigoplus_{1 \leqslant j \leqslant m} C_{e_j},$$

where C_{e_j} is the cyclic group of order e_j, $m \geqslant 2$ and $e_{j+1} \mid e_j$ for $1 \leqslant j \leqslant m-1$. With

$$H \cong \bigoplus_{1 \leqslant j \leqslant m-1} C_{e_j} \quad \text{and} \quad K \cong C_{e_m},$$

we have

$$h = \prod_{j=1}^{m-1} e_j \quad \text{and} \quad k = e_m, \quad \text{so} \quad k \mid h.$$

By Olson's theorem, S represents $0 \in G$ if $n \geqslant h+k-1$. Thus, it suffices to see that $2n-1 \geqslant g = hk$ ensures $n \geqslant h+k-1$. This is easy; for if at least one of h, k is even, we require $\frac{1}{2}hk \geqslant h+k-2$, so $(h-2)(k-2) \geqslant 0$, and if both h and k are odd, we require $(h-2)(k-2) \geqslant 1$, which conditions are satisfied because $h \geqslant k \geqslant 2$. ∎

Sequences not representing zero. Let $S = (a_i)_{i=1}^{k}$ be a sequence of k distinct elements from an abelian group $\langle G, +\rangle$, such that S does not represent 0, and let $f(k)$ denote the minimum number of elements which can be represented by S, i.e.,

$$f(k) = \min_{S,G} |\{x \in G \colon x \text{ is represented by } S\}|.$$

THEOREM 4. $f(k) \geqslant 2k-1$ *for* $k \geqslant 1$.

Proof. Clearly $f(1) = 1$. For some $k \geqslant 1$, suppose $f(k) \geqslant 2k-1$, and let $S = (a_i)_{i=1}^{k+1}$ be a $(k+1)$-sequence of distinct elements from G which does not represent 0.

Case (i). There is a term in S which is not represented by the remaining k terms. Then without loss of generality we assume a_{k+1} is such a term. The $2k-1$ elements (or more) which are represented by the first k terms of S do not include a_{k+1}; nor do they include $\sum_{i=1}^{k+1} a_i$, for otherwise the difference between this sum and some other representation of the same element is 0, contradicting the fact that S does not represent 0. Hence S represents at least $2k+1$ elements.

Case (ii). Every term in S is represented by the remaining k terms. To resolve this case we use a theorem of Moser and Scherk [3], viz.

THEOREM (Moser and Scherk). *If A, B are finite sets of elements from an abelian group $\langle G, +\rangle$, such that $0 \in A$, $0 \in B$, and $a+b = 0$, $a \in A$, $b \in B$ implies $a = 0 = b$, then $|A+B| \geqslant |A|+|B|-1$, where $A+B = \{a+b : a \in A, b \in B\}$.*

Thus, if we let $A = B = \{0, a_1, a_2, \ldots, a_{k+1}\}$, then $|A+B| \geqslant 2k+3$. Under the assumptions of case (ii), every expression of the form $2a_j$ is expressible in the form $a_j + \sum_{i=1}^{k+1} \varepsilon_i a_i$, where $\varepsilon_i \in \{0, 1\}$ for $1 \leqslant i \leqslant k+1$, and not all the ε_i are zero, but $\varepsilon_j = 0$. This shows that every element of $A+B$ other than 0 is represented by S, yielding a total of at least $2k+2$ elements represented by S. The theorem now follows by induction on k. ■

Attainment of the bound for $f(k)$ in Theorem 4, with $k = 1, 2, 3$, is shown by $1 \pmod 2$; $1, 2 \pmod 4$; $1, 3, 4 \pmod 6$.

THEOREM 5. $f(k) \geqslant 2k$ *for* $k \geqslant 4$.

Proof. The proof of Theorem 4 shows that in Case (i) if the first k terms of S represent at least $2k$ elements of G, then S represents at least $2k+2$ elements, while in Case (ii) this conclusion is invariably valid. Thus, the present theorem follows by induction on k, provided it can be shown in Case (i) that if $k = 3$ and the first 3 terms of S represent only 5 elements of G, nevertheless S represents at least 8 elements. Under these circumstances we may assume that $a_1, a_2, a_3, a_1+a_2, a_1+a_2+a_3$ are all different, and $a_1 = a_2+a_3$, $a_2 = a_1+a_3$, so $2a_3 = 0$. (It is not possible to have further independent restrictions consistent with the conditions that a_1, a_2, a_3 are distinct and do not represent 0.) Also, we may assume a_4 is not represented by a_1, a_2, a_3. Then, as before, a_4 and $a_1+a_2+a_3+a_4$ are distinct and are not represented by a_1, a_2, a_3; the same is true for $a_1+a_2+a_4$, for in particular $a_1+a_2+a_4 = a_3$ would imply

$a_1+a_2+a_3+a_4=0$, contradicting hypotheses concerning S. Thus, S represents at least 8 elements of G. The theorem follows. ■

Attainment of the bound for $f(k)$ in Theorem 5, with $k=4$, is shown by 1, 3, 4, 7 (mod 9). In general, precise evaluation of $f(k)$ is increasingly laborious, even though entirely elementary. We have shown $f(5)=13$. The proof is available as an appendix in [1]. Furthermore $f(6)\leqslant 19$, and equality seems likely. (Computations in this direction are in progress.)

Szemerédi [5] can show $f(k)\geqslant ck^2$, where c is some positive constant. On the other hand, $f(k)\leqslant[\frac{1}{2}k^2]+1$, as shown by the following two examples (where s is any positive integer);

(1) $a_i=i$ for $1\leqslant i\leqslant s$, $a_i=s^2+i$ for $s+1\leqslant i\leqslant 2s+1$ (mod $2s^2+2s+2$), where $k=2s+1$, and the number of elements represented is $\frac{1}{2}k^2+\frac{1}{2}$;

(2) $a_i=i$ for $1\leqslant i\leqslant s$, $a_i=s^2-s+i$ for $s+1\leqslant i\leqslant 2s$ (mod $2s^2+2$), where $k=2s$, and the number of elements represented is $\frac{1}{2}k^2+1$.

It is interesting to note that in all resolved cases, $f(k)$ can be achieved within the class of cyclic groups. We conjecture this to be the case for all k.

Finally we remark that our theorems perhaps carry over to non-abelian groups, but we have no results in this direction.

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