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On the Number of Unique Subgraphs of a Graph

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A subgraph H of a graph G is unique if H is not isomorphic to any other subgraph of G. The existence of a graph on n vertices having at least $2^{n^2/2-cn^{3/2}}$ unique subgraphs is proven for $c > \frac{3}{2}\sqrt{2}$ and n sufficiently large.

We will say a subgraph H of a graph G is *unique* if H is not isomorphic to any other subgraph of G. In Figure 1 we give an example of a graph and its unique subgraphs.

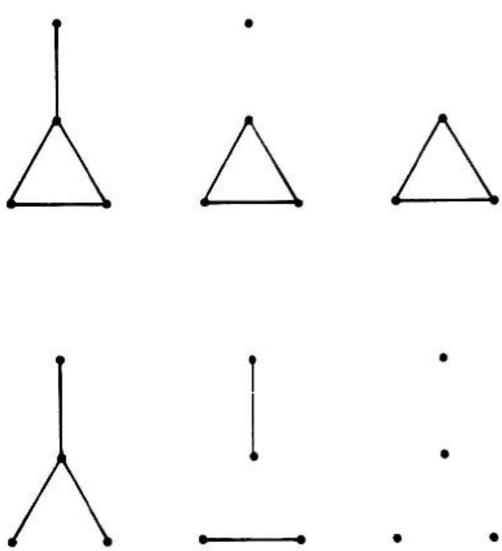

FIG. 1. A graph and its unique subgraphs.

Any graph G with edges contains at least two unique subgraphs: G itself and the graph obtained by deleting all edges of G. The complete graphs on more than one vertex have just two unique subgraphs.

We will show that, although a graph on n vertices can have at most $2^{\binom{n}{2}}$ subgraphs, still, for all large n, there are graphs with more than $2^{n^2/2-cn^{3/2}}$ unique spanning subgraphs where c is any constant greater than $\frac{3}{2}\sqrt{2}$.

In the following an *asymmetric* graph is a graph with no non-trivial automorphism and $[x]$ and $\{x\}$ have their usual meanings as largest integer not more than x and least integer not less than x, respectively. We denote by $f(n)$ the largest number of unique-subgraphs a graph on n vertices can have.

THEOREM. *$f(n) > 2^{n^2/2-cn^{3/2}}$ for $c > \frac{3}{2}\sqrt{2}$ and n sufficiently large.*

Proof. We will construct a graph G having the required number of unique subgraphs by constructing graphs A and B and then joining their vertices in a certain manner to form G.

We first construct a graph A on

$$m = \left[\frac{-1+\sqrt{8n+1}}{2}\right]$$

vertices so that the complement of A is a tree T having exactly one vertex v of valence three and such that the removal of v leaves three paths no two having the same length. Such a tree exists for $m \geqslant 7$ and hence for all $n \geqslant 28$. Clearly T, and so A, is asymmetric. For a later calculation we note that each vertex of A has valence at least $m - 4$.

Next we construct a second graph B with $n - m$ vertices by dividing these vertices as equally as possible into

$$k = \left[\frac{m}{2}\right] - 3$$

sets and joining by an edge any two vertices not in the same set. Again for future calculation we note, since there are at least

$$\left[\frac{n-m}{k}\right]$$

vertices in each set, that each vertex of B has valence at most

$$n - m - \left[\frac{n-m}{k}\right].$$

With each vertex b of B we associate a set A_b of at least $m - 2$ vertices of A. Since $\binom{m}{2} + m + 1 \geqslant n - m$, the A_b sets can be chosen to be distinct. Since also $\binom{m}{2} \leqslant n - m$ the A_b sets can be chosen to include all subsets of $m-2$ vertices of A and hence so that any vertex of A is a member of at most one more A_b set than any other vertex of A.

Now we let G be the graph consisting of A and B together with all edges joining each vertex b of B to all the vertices of A_b. If a and b are vertices of A and B, respectively, then, since there are at least $(n-m)(m-2)$ such edges, a has valence at least

$$\left[\frac{(n-m)(m-2)}{m}\right]+m-4$$

and b has valence at most

$$n-\left[\frac{n-m}{k}\right]$$

so that a has greater valence than b if

$$\left[\frac{n-m}{k}\right]>2+\left\{\frac{2n}{m}\right\}.$$

It is easy to verify that this inequality holds for $n \geqslant 1$.

If the number of edges in B is t then, since

$$\frac{n-m}{k} \leqslant \frac{2(n-m)}{m-7} \leqslant \sqrt{2n}+7,$$

we have, for $c > \frac{3}{2}\sqrt{2}$ and sufficiently large n,

$$\begin{aligned} t &\geqslant \frac{1}{2}(n-m)\left(n-m-\left\{\frac{n-m}{k}\right\}\right) > \frac{1}{2}(n-\sqrt{2n})(n-2\sqrt{2n}-8) \\ &\geqslant \frac{n^2}{2}-cn^{3/2}, \end{aligned}$$

so that the proof will be complete if we show that any subgraph of G obtained by deleting edges of B is unique.

Suppose that H is any such subgraph and that H' is another subgraph of G isomorphic to H under φ. φ must carry a vertex a of A to a vertex of A since the degree of a in H is larger than the degree in H' of any vertex of B. The restriction of φ to A is then an automorphism on A and so each vertex of A is fixed by φ since A is asymmetric. Since this is so, for each vertex b of B we must have $A_b = A_{\varphi(b)}$, which in turn requires $\varphi(b) = b$, i.e., φ is the identity and H is unique.

We note that it follows from the proof that for proper choice of the constant c we have

$$f(n) > 2^{n^2/2-3\sqrt{2}n^{3/2}-cn} \qquad \text{for} \quad n \geqslant 1.$$

It would be interesting to get a non-trivial upper bound for $f(n)$. Perhaps our lower bound is close to being best possible but we have not even proved $f(n) < 2^{n^2/2-n^{1+c}}$ for a certain $c > 0$.

The following question might also be of interest: Determine or estimate the largest $r = r(n)$ so that there is a graph on n vertices in which the removal of r or fewer edges leaves a unique subgraph.

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