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ON SOME PROBLEMS OF A STATISTICAL GROUP THEORY VII

by

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To the memory of A. RÉNYI

1. In the first paper of this series (see [**1**])[1] we proved for arbitrary small $\varepsilon > 0$ that for almost all P elements of the symmetric group S_n of n letters (i.e. with exception of $o(n!)$ elements at most)[2] the inequality

$$\exp\left\{\left(\frac{1}{2}-\varepsilon\right)\log^2 n\right\} \leq O(P) \leq \exp\left\{\left(\frac{1}{2}+\varepsilon\right)\log^2 n\right\} \quad (\exp x = e^x); \tag{1.1}$$

here $O(P)$ means of course the group theoretic order of P.[3] This is surprisingly low compared to Landau's theorem[4] according which

$$\lim_{n\to\infty} \frac{1}{\sqrt{n\log n}} \max_{P\in S_n} \log O(P) = 1. \tag{1.2}$$

Since the elements P of any fixed conjugacy class K of S_n are of the same order which might be denoted by $O(K)$, it is natural to ask what is the statistical theorem on the distribution of the orders $O(K)$ if as "equally probably events" the classes K are considered. The number of the classes K — as well known — equals to $p(n)$, the number of partitions of n for which the asymptotic formula of Hardy-Ramanujan (see [4])

$$p(n) = \big(1+o(1)\big)\frac{1}{4n\sqrt{3}}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) \tag{1.3}$$

holds. Then we state the following

THEOREM. *For arbitrarily small $\varepsilon > 0$ the inequality*

$$\exp\{(A_0-\varepsilon)\sqrt{n}\} \leq O(K) \leq \exp\{(A_0+\varepsilon)\sqrt{n}\} \tag{1.4}$$

[1] Numbers in bracket refer to the bibliography at the end of the paper.
[2] The o-sign refers to $n \to \infty$.
[3] A stronger form of (1.1) can be found in our paper [2].
[4] See LANDAU [3].

with the constant

$$A_0 = \frac{2\sqrt{6}}{\pi} \sum_{j\neq 0} \frac{(-1)^{j+1}}{3j^2+j} \sim 1{,}81 \tag{1.5}$$

holds for almost all classes K (i.e. with exception of $o(p(n))$ classes at most).

This theorem contributes again to the picture which shows that the "asymptotic structure" of the group S_n is rather transparent. Harmonising this theorem and (1.1) we can conclude that (1.1) is caused by the elements P of a "few but populous" classes K.

For further remarks see 13.

2. For the proof of our theorem we shall use for $\operatorname{Re} z > 0$ the function

$$f(z) \overset{\text{def}}{=} \sum_{n=0}^{\infty} p(n)\, e^{-nz} = \prod_{\nu=1}^{\infty} \frac{1}{1-e^{-\nu z}} \tag{2.1}$$

notably the functional equation

$$f(z) = \sqrt{\frac{z}{2\pi}} \exp\left\{-\frac{z}{24}+\frac{\pi^2}{6z}\right\} f\left(\frac{4\pi^2}{z}\right). \tag{2.2}$$

This gives in all angles $|\operatorname{arc} z| \leq \alpha < \frac{\pi}{2}$ the relation

$$f(z) = \big(1+o(1)\big)\sqrt{\frac{z}{2\pi}}\exp\left(\frac{\pi^2}{6z}\right) \qquad \text{for } z\to 0; \tag{2.3}$$

further[5] for $0 < x \leq 1 \quad (z = x+iy)$

$$c\sqrt{x}\exp\left(\frac{\pi^2}{6x}\right) \leq f(x) \leq c\sqrt{x}\exp\left(\frac{\pi^2}{6x}\right). \tag{2.4}$$

From (2.1) we get easily for $x \geq 1$

$$1+e^{-x} \leq f(x) \leq 1+ce^{-x}; \tag{2.5}$$

this and (2.4) give for each $y \geq 1$ the rough but useful inequality

$$c\sqrt{\frac{x}{y}}\exp\left(\frac{\pi^2}{6x}\right) \leq f(x) \leq c\sqrt{x}\exp\left(\frac{\pi^2}{6x}\right). \tag{2.6}$$

valid for $0 < x \leq y$. We remind also the "Pentagonalzahlsatz" of Euler-Legendre

$$\prod_{\nu=1}^{\infty}(1-e^{-\nu z}) = \sum_{j=-\infty}^{\infty}(-1)^j \exp\left(-\frac{3j^2+j}{2}z\right). \tag{2.7}$$

[5] c mean throughout this paper unspecified positive numerical constants, not necessarily the same in different occurrences.

3. We shall need some lemmata.

Lemma I. *For* $0 \leqq \gamma \leqq 1$ *and real* $x > 0$ *the inequality*

$$\varphi_\gamma(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} n^\gamma p(n)\, e^{-nx} < c x^{\frac{1}{2}-2\gamma} \exp\left(\frac{\pi^2}{6x}\right)$$

holds.

For the proof we remark first that the functional equation (2.2) gives in connection with (2.3) for $x \to +0$ easily

$$-f'(x) = \sum_{n=1}^{\infty} np(n)\, e^{-nx} = \bigl(1 + o(1)\bigr) \frac{\pi^{3/2}}{6\sqrt{2}} x^{-\frac{3}{2}} \exp\left(\frac{\pi^2}{6x}\right). \tag{3.1}$$

Then Hölder's inequality gives

$$\varphi_\gamma(x) = \sum_{n=1}^{\infty} \{n^\gamma p(n)^\gamma (e^{-nx})^\gamma\}\, \{p(n)^{1-\gamma} (e^{-nx})^{1-\gamma}\} \leqq$$

$$\leqq \Bigl(\sum_{n=1}^{\infty} np(n)\, e^{-nx}\Bigr)^\gamma \Bigl(\sum_{n=1}^{\infty} p(n)\, e^{-nx}\Bigr)^{1-\gamma}$$

which is owing to (3.1) and (2.3)

$$< c x^{\frac{1}{2}-2\gamma} \exp\left(\frac{\pi^2}{6x}\right)$$

indeed.

Next let us consider

$$H_1(n) \overset{\text{def}}{=} \sum_{K} \log O(K). \tag{3.2}$$

Denoting

$$A_1 \overset{\text{def}}{=} \sum_{j \neq 0} \frac{(-1)^{j+1}}{3\, j^2 + j} \tag{3.3}$$

we assert the

Lemma II. *For* $n \to \infty$ *the relation*

$$H_1(n) = \bigl(1 + o(1)\bigr) \frac{2\, A_1 \sqrt{6}}{\pi} \sqrt{n}\; p(n)$$

holds.

4. For the proof we remind first that to each conjugacy class K we have a uniquely determined partition Q of

$$\begin{gathered} n = m_1 n_1 + m_2 n_2 + \ldots + m_k n_k \\ 1 \leq n_1 < n_2 < \ldots < n_k \end{gathered} \tag{4.1}$$

so that

$$O(K) = [n_1, n_2, \ldots, n_k]. \tag{4.2}$$

Hence

$$H_1(n) = \sum_Q \log [n_1, n_2, \ldots, n_k]. \tag{4.3}$$

Thus if $q_1, q_2, \ldots$ denote always primes, we have[6]

$$H_1(n) = \sum_Q \sum_{\substack{q, \alpha \\ q^\alpha \| [n_1, \ldots, n_k]}} \alpha \log q \tag{4.4}$$

and also obviously

$$H_1(n) = \sum_{\substack{q, \alpha \\ q^\alpha \leq n}} \alpha \log q \, Z(q, \alpha) \tag{4.5}$$

with

$$Z(q, \alpha) = \sum_Q{}' 1 \tag{4.6}$$

where the summation is to be extended over all Q-partitions where no summands are divisible by $q^{\alpha+1}$ but at least one summand is divisible by q^α. Since the number of partitions with no summands divisible by $q^{\alpha+1}$ is

$$= \text{coeffs. } e^{-nz} \text{ in } \prod_{q^{\alpha+1} \nmid \nu} \frac{1}{1 - e^{-\nu z}}$$

or using the representation (2.1)

$$= \text{coeffs. } e^{-nz} \text{ in } \frac{f(z)}{f(q^{\alpha+1} z)};$$

hence

$$Z(q, \alpha) = \text{coeffs. } e^{-nz} \text{ in } f(z) \left(\frac{1}{f(q^{\alpha+1} z)} - \frac{1}{f(q^\alpha z)} \right) \tag{4.7}$$

and with the notation

$$\left[\frac{\log n}{\log q} \right] = \alpha_q$$

from (4.5)

$$H_1(n) = \sum_{q \leq n} \log q \sum_{1 \leq \alpha \leq \alpha_q} \alpha \, Z(q, \alpha) =$$

$$= \text{coeffs. } e^{-nz} \text{ in } f(z) \sum_{q \leq n} \log q \sum_{\alpha=1}^{\alpha_q} \alpha \left(\frac{1}{f(q^{\alpha+1} z)} - \frac{1}{f(q^\alpha z)} \right).$$

[6] The symbol $q^\alpha \| m$ means that $q^\alpha | m$ but $q^{\alpha+1} \nmid m$.

The internal sum is

$$-\frac{1}{f(qz)}-\frac{1}{f(q^2z)}-\dots-\frac{1}{f(q^{\alpha_q}z)}+\frac{\alpha_q}{f(q^{\alpha_q+1}z)}$$

and since the term $f(q^{\alpha_q+1}z)$ contributes to the coefficient of e^{-nz} owing to (2.7) only through its constant term we get

$$H_1(n)=\text{coeffs.}\,e^{-nz}\text{ in }f(z)\sum_{q\le n}\log q\sum_{\alpha=1}^{\alpha_q}\left(1-\frac{1}{f(q^\alpha z)}\right)$$

and also

$$H_1(n)=\text{coeffs.}\,e^{-nz}\text{ in }f(z)\sum_{q}\log q\sum_{\alpha=1}^{\infty}\left(1-\frac{1}{f(q^\alpha z)}\right)$$

and using finally the representation in (2.7)

$$H_1(n)=\text{coeffs.}\,e^{-nz}\text{ in }f(z)\sum_{q}\log q\sum_{\alpha=1}^{\infty}\sum_{j\ne 0}(-1)^{j+1}\exp\left(-\frac{3j^2+j}{2}q^\alpha z\right). \tag{4.8}$$

5. Next we have to investigate the triple sum in (4.8). Using the Mellin integral formula this is as easy to see for $\operatorname{Re} z>0$

$$=\frac{1}{2\pi i}\int_{(2)}\frac{\Gamma(s)}{z^s}\left(\sum_{j\ne 0}\frac{(-1)^{j+1}}{\left(\frac{3j^2+j}{2}\right)^s}\right)\left(\sum_q\sum_{\alpha=1}^{\infty}\frac{\log q}{q^{\alpha s}}\right)ds=$$

$$=-\frac{1}{2\pi i}\int_{(2)}\frac{\Gamma(s)}{z^s}\frac{\zeta'}{\zeta}(s)\left(\sum_{j\ne 0}\frac{(-1)^{j+1}}{\left(\frac{3j^2+j}{2}\right)^s}\right)ds.$$

Usual contour-integration technique and elementary properties of $\zeta(s)$ give that this sum is

$$=\big(1+o(1)\big)\frac{2A_1}{z} \tag{5.1}$$

(see (3.3)) if z tends to 0 from any angle from the right half plane. Together with (2.3) this gives

$$\sum_{n=1}^{\infty}H_1(n)\,e^{-nz}=\big(1+o(1)\big)\frac{A_1}{\sqrt{2\pi z}}\exp\left(\frac{\pi^2}{6z}\right) \tag{5.2}$$

if only $z\to 0$ in any angle from the right half plane. Since $H_1(n)$ is non-decreasing, the coefficients of

$$\begin{aligned}F(z)&\overset{\text{def}}{=}H_1(z)\,e^{-z}+\sum_{n=2}^{\infty}\big(H_1(n)-H_1(n-1)\big)e^{-nz}=\\&=(1-e^{-z})\sum_{n=1}^{\infty}H_1(n)\,e^{-nz}\end{aligned} \tag{5.3}$$

are nonnegative and from (5.2) we get

$$F(z) = \big(1 + o(1)\big)\frac{2\,A_1}{\sqrt{2\pi}}\sqrt{z}\,\exp\left(\frac{\pi^2}{6z}\right). \tag{5.4}$$

Hence the general Tauberian theorem of INGHAM (see INGHAM [5]), together with (1.3) completes the proof of Lemma II.

Next we consider

$$H_2(n) \overset{\text{def}}{=} \sum_K \log^2 O(K). \tag{5.5}$$

Then we assert the

LEMMA III. *For $n \to \infty$ the relation*

$$H_2(n) = \big(1 + o(1)\big)\left(\frac{2\,A_1\sqrt{6}}{\pi}\right)^2 n\,p(n)$$

holds.

6. For the proof we remark first that owing to (4.2)

$$H_2(n) = \sum_Q \log^2 [n_1, n_2, \ldots, n_k]. \tag{6.1}$$

As before

$$\begin{aligned} H_2(n) = \sum_Q \sum_{q^\alpha \| [n_1,\ldots,n_k]} \alpha^2 \log^2 q \,+ \\ + \sum_Q \sum_{\substack{q_1 \neq q_2 \\ q_1^{\alpha_1} \| [n_1,\ldots,n_k] \\ q_2^{\alpha_2} \| [n_1,\ldots,n_k]}} \alpha_1\,\alpha_2 \log q_1 \log q_2 = H_2'(n) + H_2''(n). \end{aligned} \tag{6.2}$$

Since $q_\nu^{\alpha_\nu} \leq n$, we have owing to (4.4) and Lemma II

$$H_2'(n) \leq \log n \sum_Q \sum_{q \| [n_1,\ldots,n_k]} \alpha \log q = \log n\, H_1(n) < c\sqrt{n}\log n \cdot p(n). \tag{6.3}$$

Hence it suffices to investigate

$$H_2''(n) = \sum_{\substack{q_1 \neq q_2 \\ q_1^{\alpha_1} \leq n,\, q_2^{\alpha_2} \leq n \\ \alpha_1 \geq 1,\, \alpha_2 \geq 1}} a_1\,\alpha_2 \log q_1 \log q_2\, R(q_1, q_2, \alpha_1, \alpha_2) \tag{6.4}$$

where

$$R(q_1, q_2, \alpha_1, \alpha_2) = \sum_Q{}^{*}\, 1$$

where the summation is to be extended to all Q-partitions for which no summand is divisible neither by $q_1^{\alpha_1+1}$ nor by $q_2^{\alpha_2+1}$ but some summand is divisible by

$q_1^{\alpha_1}$ and some by $q_2^{\alpha_2}$. The number of Q-partitions satisfying the first two requirements is

$$= \text{coeffs.}\, e^{-nz} \text{ in } \prod_{\substack{q_1^{\alpha_1+1} \nmid \nu \\ q_2^{\alpha_2+1} \nmid \nu}} \frac{1}{1-e^{-\nu z}} = \text{coeffs.}\, e^{-nz} \text{ in } \frac{f(z)\, f(q_1^{\alpha_1+1} q_2^{\alpha_2+1} z)}{f(q_1^{\alpha_1+1} z)\, f(q_2^{\alpha_2+1} z)}$$

with the notation (2.1). We have to subtract from this the number of those Q-partitions where either no summand is divisible by $q_1^{\alpha_1}$, or no summand is divisible by $q_2^{\alpha_2}$. The number of these partitions is

$$\text{coeffs.}\, e^{-nz} \text{ in } \left\{\frac{f(z)}{f(q_1^{\alpha_1} z)} + \frac{f(z)}{f(q_2^{\alpha_2} z)} - \frac{f(z)\, f(q_1^{\alpha_1} q_2^{\alpha_2} z)}{f(q_1^{\alpha_1} z)\, f(q_2^{\alpha_2} z)}\right\}$$

and hence

$$R(q_1, q_2, \alpha_1, \alpha_2) = \text{coeffs.}\, e^{-nz} \text{ in } \left\{\frac{f(q^{\alpha_1+1} q^{\alpha_2+1} z)}{f(q_1^{\alpha_2+1} z)\, f(q_2^{\alpha_2+1} z)} - \right.$$

$$\left. - \frac{1}{f(q_1^{\alpha_1} z)} - \frac{1}{f(q_2^{\alpha_2} z)} + \frac{f(q_1^{\alpha_1} q_2^{\alpha_2} z)}{f(q_1^{\alpha_1} z)\, f(q_2^{\alpha_2} z)}\right\} f(z). \tag{6.6}$$

For later aims we write the expression in the bracket in the form

$$\left(1 - \frac{1}{f(q_1^{\alpha_1} z)}\right)\left(1 - \frac{1}{f(q_2^{\alpha_2} z)}\right) + \frac{f(q_1^{\alpha_1} q_2^{\alpha_2} z) - 1}{f(q_1^{\alpha_1} z)\, f(q_2^{\alpha_2} z)} +$$

$$+ \frac{f(q_1^{\alpha_1+1} q_2^{\alpha_2+1} z) - 1}{f(q_1^{\alpha_1+1} z)\, f(q_2^{\alpha_2+1} z)} - \left(1 - \frac{1}{f(q_1^{\alpha_1+1} z)\, f(q_2^{\alpha_2+1} z)}\right). \tag{6.7}$$

Then $H_2''(n)$ in the form (6.4) can be represented as

$$H_2''(n) = L_1(n) + L_2(n) + L\ (n) - L_4(n) \tag{6.8}$$

where

$$L_1(n) \stackrel{\text{def}}{=} \text{coeffs.}\, e^{-nz} \text{ in } f(z) \left\{\sum_{\substack{q_1^{\alpha_1} \leq n \\ \alpha_1 \geq 1}} \sum_{\substack{q_2^{\alpha_2} \leq n \\ \alpha_2 \geq 1 \\ q_1 \neq q_2}} \alpha_1 \alpha_2 \log q_1 \log q_2 \left(1 - \frac{1}{f(q_1^{\alpha_1} z)}\right)\left(1 - \frac{1}{f(q_2^{\alpha_2} z)}\right)\right\}$$

$$L_2(n) \stackrel{\text{def}}{=} \text{coeffs.}\, e^{-nz} \text{ in } f(z) \left\{\sum \sum_{\substack{q_1 \neq q_2 \\ q_1^{\alpha_1} q_2^{\alpha_2} \leq n}} \alpha_1 \alpha_2 \log q_1 \log q_2 \frac{f(q_1^{\alpha_1} q_2^{\alpha_2} z) - 1}{f(q_{11}^{\alpha} z)\, f(q_2^{\alpha_2} z)}\right\}$$

(6.9)

$$L_3(n) \stackrel{\text{def}}{=} \text{coeffs.}\, e^{-nz} \text{ in } f(z) \left\{\sum \sum_{\substack{q_1^{\alpha_1+1} q_2^{\alpha_2+1} \leq n \\ q_1 \neq q_2}} \alpha_1 \alpha_2 \log q_1 \log q_2 \frac{f(q_1^{\alpha_1+1} q_2^{\alpha_2+1} z) - 1}{f(q_1^{\alpha_1+1} z)\, f(q_2^{\alpha_2+1} z)}\right.$$

$$L_4(n) \stackrel{\text{def}}{=} \text{coeffs.}\, e^{-nz} \text{ in}$$

$$f(z)\left\{\sum_{\substack{q_1^{\alpha_1+1}\le n\\ \alpha_1\ge 1}}\ \sum_{\substack{q_2^{\alpha_2+1}\le n\\ \alpha_2\ge 1\\ q_1\ne q_2}} \alpha_1\alpha_2\log q_1\log q_2\left(1-\frac{1}{f(q_1^{\alpha_1+1}z)f(q_2^{\alpha_2+1}z)}\right)\right\}$$

(taking into account e. g. in $L_2(n)$ that terms in the sum with $q_1^{\alpha_1}q_2^{\alpha_2}>n$ do not contribute to coeffs. e^{-nz} at all).

7. The easiest is to deal with $L_4(n)$. Using (2.1) and (2.7) we get the representation

$$\begin{aligned} L_4(n) = &\sum_{\substack{q_1^{\alpha_1+1}\le n\\ \alpha_1\ge 1}}\ \sum_{\substack{q_2^{\alpha_2+1}\le n\\ \alpha_2\ge 1\\ q_1\ne q_2}} \alpha_1\alpha_2\log q_1\log q_2 \sum_{j_1^2+j_2^2>0}(-1)^{j_1+j_2+1}\times \\ &\times p\left(n-\frac{3j_1^2+j_1}{2}q_1^{\alpha_1+1}-\frac{3j_2^2+j_2}{2}q_2^{\alpha_2+1}\right), \end{aligned} \tag{7.1}$$

or — using the monotonicity of $p(m)$ in m —

$$|L_4(n)|\le c\log^4 n\sum_{q_1\le\sqrt{n}}\ \sum_{q_2\le\sqrt{n}}\ \sum_{\substack{q_1^2j_1^2+q_2^2j_2^2\le n\\ j_1^2+j_2^2>0}} p(n-q_1^2j_1^2-q_2^2j_2^2)=c\log^4 n\big(L_4'(n)+L_4''(n)\big), \tag{7.2}$$

where

$$\begin{aligned} L_4'(n) &\overset{\text{def}}{=} \sum_{q_1\le\sqrt{n}}\ \sum_{q_2\le\sqrt{n}}\ \sum_{\substack{q_1^2j_1^2+q_2^2j_2^2\le 100\sqrt{n}\log n\\ j_1^2+j_2^2>0}} p(n-q_1^2j_1^2-q_2^2j_2^2)\\ L_4''(n) &\overset{\text{def}}{=} \sum_{q_1\le\sqrt{n}}\ \sum_{q_2\le\sqrt{n}}\ \sum_{100\sqrt{n}\log n\le q_1^2j_1^2+q_2^2j_2^2\le n} p(n-q_1^2j_1^2-q_2^2j_2^2)\,. \end{aligned} \tag{7.3}$$

Since from (1.3) for $1\le m\le n-1$

$$\begin{aligned} &\frac{p(n-m)}{p(n)}<cn\exp\left\{\frac{2\pi}{\sqrt{6}}\left(\sqrt{n-m}-\sqrt{n}\right)\right\}=\\ &cn\exp\left(-\frac{m}{\sqrt{n-m}+\sqrt{n}}\right)<cn\exp\left(-\frac{m}{2\sqrt{n}}\right), \end{aligned} \tag{7.4}$$

we get

$$| L_4''(n)<cn\,p(n)\sum_{q_1\le\sqrt{n}}\sum_{q_2\le\sqrt{n}}\ \sum_{j_1^2q_1^2+j_2^2q_2^2>100\sqrt{n}\log n}\exp\left(-\frac{j_1^2q_1^2+j_2^2q_2^2}{2\sqrt{n}}\right)$$

which is trivially

$$<c\,n^2p(n)\sum_{m>100\sqrt{n}\log n} m\exp\left(-\frac{m}{2\sqrt{n}}\right)=o\big(p(n)\big). \tag{7.5}$$

For $L_4'(n)$ — since instead of (7.4) we have now

$$\frac{p(n-m)}{p(n)} < 1 -$$

we get the upper bound

$$2\,p(n) \sum_{q_1 \leq \sqrt{n}} \sum_{q_2 \leq \sqrt{n}} \sum_{\substack{j_1^2 q_1^2 + j_2^2 q_2^2 < 100\sqrt{n}\log n \\ j_1 \neq 0}} 1 = cp(n)\Big\{\sqrt{n} \sum_{q_1 \leq \sqrt{n}} \sum_{1 \leq |j_1| \leq \frac{10}{q_1} n^{1/4}\log n} + \\ + \Big(\sum_{q \leq \sqrt{n}} \sum_{1 \leq j \leq \frac{10}{q} n^{1/4}\log n} 1\Big)^2\Big\} < cn^{3/4}\, p(n) \log^6 n.$$

This, (7.5) and (7.2) give indeed

$$L_4(n) < cn^{3/4}\, p(n) \log^{10} n\,. \tag{7.6}$$

8. For $L_2(n)$ we need a different treatment. We observe first that the coefficient of e^{-nz} in

$$\frac{f(z)\left(f(q_1^{\alpha_1} q_2^{\alpha_2} z) - 1\right)}{f(q_1^{\alpha_1} z) f(q_2^{\alpha_2} z)}$$

is the same as in

$$\frac{f(z) f(q_1^{\alpha_1} q_2^{\alpha_2} z)}{f(q_1^{\alpha_1} z) f(q_2^{\alpha_2} z)} \left(1 - \frac{1}{f(q_1^{\alpha_1} q_2^{\alpha_2} z)}\right)$$

and hence owing to the representation (2.7) its absolute value cannot exceed

$$\sum_{j \neq 0} \left| \text{coeffs. } \exp\left\{-\left(n - \frac{3\,j^2 + j}{2} q_1^{\alpha_1} q_2^{\alpha_2} z\right)\right\} \text{ in } \frac{f(z) f(q_1^{\alpha_1} q_2^{\alpha_2} z)}{f(q_1^{\alpha_1} z) f(q_2^{\alpha_2} z)} \right|.$$

Further we may observe that the coefficient of e^{-mz} in $\dfrac{f(z) f(q_1^{\alpha_1} q_2^{\alpha_2} z)}{f(q_1^{\alpha_1} z) f(q_2^{\alpha_2} z)}$ — being the counting number of *certain* partitions of m — cannot exceed $p(m)$ (and is nonnegative). Thus defining

$$L_2'(n) \overset{\text{def}}{=} \text{coeffs. } e^{-nz} \text{ in} \left\{ \sum_{\substack{q_1 \neq q_2 \\ 100\sqrt{n}\log n \leq q_1^{\alpha_1} q_2^{\alpha_2} \leq n}} \sum \alpha_1 \alpha_2 \log q_1 \log q_2 \frac{f(z)\left(f(q_1^{\alpha_1} q_2^{\alpha_2} z) - 1\right)}{f(q_1^{\alpha_1} z) f(q_2^{\alpha_2} z)} \right\} \tag{8.1}$$

we have

$$|L_2'(n)| \leq \log^2 n \sum_{\substack{100\sqrt{n}\log n \leq q_1^{\alpha_1} q_2^{\alpha_2} \leq n \\ q_1 \neq q_2}} \sum \sum_{j \neq 0} p\left(n - \frac{3\,j^2 + j}{2} q_1^{\alpha_1} q_2^{\alpha_2}\right).$$

Using further (7.4) we get

$$|L_2'(n)| \leq cnp(n) \log^2 n \sum_{\substack{100\sqrt{n}\log n \leq q_1^{\alpha_1} q_2^{\alpha_2} \leq n \\ q_1 \neq q_2}} \sum \sum_{j \neq 0} \exp\left\{-\frac{(3\,j^2 + j)\, q_1^{\alpha_1} q_2^{\alpha_2}}{4\sqrt{n}}\right\}. \tag{8.2}$$

Since the inner sum cannot exceed

$$c \exp\left\{-\frac{q_1^{\alpha_1} q_2^{\alpha_2}}{4\sqrt{n}}\right\}$$

we get further roughly

$$(8.3) \qquad |L_2'(n)| < cnp(n)\log^2 n \sum_{\substack{100\sqrt{n}\log n \le q_1^{\alpha_1} q_2^{\alpha_2} \le n \\ q_1 \ne q_2}} \exp\left(-\frac{q_1^{\alpha_1} q_2^{\alpha_2}}{4\sqrt{n}}\right) < cp(n).$$

Putting

$$(8.4) \qquad L_2''(n) = \sum_{\substack{q_1^{\alpha_1} q_2^{\alpha_2} \le 100\sqrt{n}\log n \\ q_1 \ne q_2}} \sum \text{coeffs. } e^{-nz} \text{ in } \frac{f(z) f(q_1^{\alpha_1} q_2^{\alpha_2} z)}{f(q_1^{\alpha_1} z) f(q_2^{\alpha_2} z)}$$

and

$$(8.5) \qquad L_2'''(n) = \sum_{\substack{q_1^{\alpha_1} q_2^{\alpha_2} \le 100\sqrt{n}\log n \\ q_1 \ne q_2}} \sum \left| \text{coeffs. } e^{-nz} \text{ in } \frac{f(z)}{f(q_1^{\alpha_1} z) f(q_2^{\alpha_2} z)} \right|$$

we get easily from (6.9), (8.1), (8.3), (8.4) and (8.5)

$$(8.6) \qquad |L_2(n)| < c\{p(n) + \log^2 n\,(L_2''(n) + L_2'''(n))\}.$$

9. In order to estimate $L_2''(n)$ let

$$(9.1) \qquad x_0 = \frac{\pi}{\sqrt{6n}}.$$

Since — as mentioned — the coefficients of $\frac{f(z) f(q_1^{\alpha_1} q_2^{\alpha_2} z)}{f(q_1^{\alpha_1} z) f(q_2^{\alpha_2} z)}$ are nonnegative; we have

$$L_2''(n) e^{-nx_0} \le \sum_{m=1}^{\infty} L_2''(m) e^{-mx_0} = \sum_{\substack{q_1^{\alpha_1} q_2^{\alpha_2} \le 100\sqrt{n}\log n \\ q_1 \ne q_2}} \sum \frac{f(x_0) f(q_1^{\alpha_1} q_2^{\alpha_2} x_0)}{f(q_1^{\alpha_1} x_0) f(q_2^{\alpha_2} x_0)}$$

or using the upper bound in (2.6)

$$L_2''(n) < c \sum_{\substack{q_1^{\alpha_1} q_2^{\alpha_2} \le 100\sqrt{n}\log n \\ q_1 \ne q_2}} \sqrt{\frac{q_1^{\alpha_1} q_2^{\alpha_2}}{n}} \exp\left(\frac{\pi\sqrt{n}}{\sqrt{6}}\right) \frac{\exp\left\{\frac{\pi\sqrt{n}}{\sqrt{6}}\left(1 + \frac{1}{q_1^{\alpha_1} q_2^{\alpha_2}}\right)\right\}}{f(q_1^{\alpha_1} x_0) f(q_2^{\alpha_2} x_0)}.$$

Using further the lower bound in (2.6) (with $y = 100 \log n$) we obtain

$$(9.2) \quad L_2''(n) < c \exp\left(\frac{\pi\sqrt{n}}{6}\right) \log^2 n \sum_{\substack{q_1^{\alpha_1} q_2^{\alpha_2} \le 100\sqrt{n}\log n \\ q_1 \ne q_2 \\ q_1^{\alpha_1} \ge q_2^{\alpha_2}}} \sum \exp\left\{\frac{\pi\sqrt{n}}{6}\left(1 - \frac{1}{q_1^{\alpha_1}}\right)\left(1 - \frac{1}{q_2^{\alpha_2}}\right)\right\}.$$

The part of this last sum corresponding to

$$q_1^{\alpha_1} < \frac{\sqrt{n}}{100 \log n} \tag{9.3}$$

cannot exceed

$$\begin{aligned} c \log^2 n \cdot \exp\left(\frac{\pi}{\sqrt{6}} \sqrt{n}\right) \sum_{q_2^{\alpha_1} \leq \frac{\sqrt{n}}{100 \log n}} \sum_{q_1^{\alpha_1} \leq \frac{\sqrt{n}}{100 \log n}} \exp\left\{\frac{\pi \sqrt{n}}{\sqrt{6}} \left(1 - \frac{1}{q_1^{\alpha_1}}\right)\right\} < \\ < c \log^2 n \cdot \frac{\sqrt{n}}{100 \log n} \exp\left(\frac{2\pi}{\sqrt{6}} \sqrt{n}\right) \cdot \frac{\sqrt{n}}{100 \log n} \cdot \exp\left(-\frac{\pi}{\sqrt{6}} \cdot 100 \log n\right) < cp(n) \end{aligned} \tag{9.4}$$

using (1.3). As to the remaining sum in (9.2), i.e. for the range

$$\frac{\sqrt{n}}{100 \log n} \leq q_1^{\alpha_1}, \quad q_1^{\alpha_1} q_2^{\alpha_2} \leq 100 \sqrt{n} \log n$$

we have

$$q_2^{\alpha_2} \leq 10^4 \log^2 n$$

i.e. the corresponding sum cannot exceed

$$\begin{aligned} c \log^2 n \cdot \exp\left(\frac{\pi \sqrt{n}}{\sqrt{6}}\right) \sum_{q_1^{\alpha_1} \leq 100 \sqrt{n} \log n} \sum_{q_2^{\alpha_2} \leq 10^4 \log^2 n} \exp\left\{\frac{\pi}{\sqrt{6}} \sqrt{n} \left(1 - \frac{1}{q_2^{\alpha_2}}\right)\right\} < \\ < c \sqrt{n} \log^5 n \cdot \exp\left(\frac{2\pi}{\sqrt{6}} \sqrt{n} - \frac{\pi}{\sqrt{6}\, 10^4} \frac{\sqrt{n}}{\log^2 n}\right) < cp(n). \end{aligned} \tag{9.5}$$

Thus from (9.4) and (9.5) we got

$$L_2''(n) < cp(n). \tag{9.6}$$

In order to show

$$L_2(n) < cn^{3/4} p(n) \log^{10} n \tag{9.7}$$

it will be (amply) sufficient owing to (8.6) to show that

$$L_2'''(n) < cn^{3/4} p(n) \log n. \tag{9.8}$$

10. To do so we have first from (8.5)

$$| L_2'''(n) \leq 2 \sum_{\substack{q_1^{\alpha_1} q_2^{\alpha_2} \leq 100 \sqrt{n} \log n \\ q_1 \neq q_2,\ q_1^{\alpha_1} \leq q_2^{\alpha_2}}} \left| \text{coeffs. } e^{-nz} \text{ in } \frac{f(z)}{f(q_1^{\alpha_1} z) f(q_2^{\alpha_2} z)} \right|$$

and using the observation that

$$(0 \leq) \text{ coeffs. } e^{-mz} \text{ in } \frac{f(z)}{f(q_1^{\alpha_1} z)} \leq p(m)$$

and also (2.7) we get

$$|L_2'''(n)| \leq 2 \sum_{q_2^{\alpha_2} \leq 10n^{1/4}\log n} \; \sum_{\substack{q_2^{\alpha_2} \leq q_1^{\alpha_1} \leq \frac{100\sqrt{n}\log n}{q_2^{\alpha_2}} \\ q_1 \neq q_2}} \; \sum_j p\left(n - \frac{3j^2+j}{2} q_1^{\alpha_1}\right) \leq$$

$$\leq 2 \sum_{q_2^{\alpha_2} \leq 10n^{1/4}\log n} \frac{100\sqrt{n}\log n}{q_2^{\alpha_2}} \sum_j p(n - j^2 q_2^{\alpha_2})$$

which cannot exceed

$$c\sqrt{n}\log n \Big\{ \sum_{q_2^{\alpha_2} \leq 10n^{1/4}\log n} q_2^{-\alpha_2} \sum_{|j|^2 > \frac{100\sqrt{n}\log n}{q_2^{\alpha_2}}} p(n - q_2^{\alpha_2} j^2) +$$

(10.1)

$$+ \sum_{q_2^{\alpha_2} \leq 10n^{1/4}\log n} q_2^{-\alpha_2} \sum_{|j|^2 \leq \frac{100\sqrt{n}\log n}{q_2^{\alpha_2}}} p(n) \Big\}.$$

The last double sum is evidently

$$cn^{1/4}\, p(n) \log n. \tag{10.2}$$

Using further (7.4) we get for the first double-sum in (10.1) the upper bound

$$< cnp(n) \sum_{q_2^{\alpha_2} \leq 10n^{1/4}\log n} q_2^{-\alpha_2} \sum_{j > \frac{10n^{1/4}\sqrt{\log n}}{q_2^{\frac{1}{2}\alpha_2}}} \exp\left(-\frac{j^2 q_2^{\alpha_2}}{2\sqrt{n}}\right) < c\frac{p(n)}{n^2}. \tag{10.3}$$

(10.2) and (10.3) give with (10.1)

$$L_2'''(n) < cn^{3/4}\, p(n) \log n$$

indeed as asserted in (9.8). Hence (9.7) is proved. The proof of

$$L_3(n) < cn^{3/4}\, p(n) \log^{10} n \tag{10.4}$$

can be done mutatis mutandis. Hence for $\nu = 2, 3, 4$ the inequality

$$|L_\nu(n)| < cp(n)\, n^{3/4} \log n \tag{10.5}$$

holds.

11. In order to complete the proof of Lemma II we have to investigate $L_1(n)$. From (6.9)

$$L_1(n) = \text{coeffs. } e^{-nz} \text{ in } f(z) \Big\{ \sum_{\substack{q^\alpha \leq n \\ \alpha \geq 1}} \alpha \log q \left(1 - \frac{1}{f(q^\alpha z)}\right) \Big\}^2 -$$

(11.1)

$$- \text{coeffs. } e^{-nz} \text{ in } f(z) \sum_{\substack{q^\alpha \leq n \\ \alpha \geq 1}} \alpha^2 \log^2 q \left(1 - \frac{1}{f(q^\alpha z)}\right)^2 \overset{\text{def}}{=} L_1'(n) - L_1''(n).$$

Let us consider first $L_1''(n)$. Writing

$$f(z)\left(1-\frac{1}{f(q^\alpha z)}\right)^2=\left(f(z)-\frac{f(z)}{f(q^\alpha z)}\right)\left(1-\frac{1}{f(q^\alpha z)}\right)$$

and observing that

$$(0\leqq)\ \text{coeffs.}\ e^{-mz}\ \text{in}\ \left(f(z)-\frac{f(z)}{f(q^\alpha z)}\right)\leqq p(m)$$

we get — using also (2.7) —

$$|L_1''(n)|\leqq \log^2 n\sum_{\substack{q^\alpha\leqq n\\ \alpha\geqq 1}}\sum_{j\neq 0}p\left(n-\frac{3j^2+j}{2}q^\alpha\right) \tag{11.2}$$

which is analogously as before

$$<c\Big(p(n)+\log^2 n\sum_{q^\alpha\leqq 100\sqrt{n}\log n}\ \sum_{\substack{j\neq 0\\ |j|\leqq 10\frac{n^{1/4}\log n}{q^{\alpha/2}}}}p(n-j^2q^\alpha)\Big)$$

$$<cp(n)\log^2 n\sum_{q^\alpha\leqq 100\sqrt{n}\log n}\frac{n^{1/4}\log n}{q^{\alpha/2}}$$

i.e.

$$|\,L_1''(n)\,|\ cp(n)\sqrt{n}\log^3 n. \tag{11.3}$$

As to $L_1'(n)$ we have for $\operatorname{Re} z>0$

$$\sum_{n=1}^{\infty}L_1'(n)\,e^{-nz}=f(z)\left\{\sum_{q,\alpha}\alpha\log q\left(1-\frac{1}{f(q^\alpha z)}\right)\right\}^2. \tag{11.4}$$

Using the reasoning of **5** and also (5.1) and (2.3) we see that the right side of (11.4)

$$=\big(1+o(1)\big)\frac{4A_1^2}{\sqrt{2\pi}}z^{-\frac{3}{2}}\exp\left(\frac{\pi^2}{6z}\right) \tag{11.5}$$

if z tends to 0 in any angle from the right half-plane. Putting

$$h(n)=L_1''(n)+L_2(n)+L_3(n)-L_4(n).$$

we get from (11.3) and (10.5)

$$|\,h(n)\,|<cp(n)\,n^{4/5} \tag{11.6}$$

and thus for $z=x+iy$

$$\Big|\sum_{n=1}^{\infty}h(n)\,e^{-nz}\Big|<c\sum_{n=1}^{\infty}n^{4/5}\,p(n)\,e^{-nx}.$$

Using Lemma I the right side is

$$< c\,x^{-\frac{11}{10}} \exp\left(\frac{\pi^2}{6x}\right);$$

this and (11.5) result owing to (6.8) and (6.3) that

$$\sum_{n=1}^{\infty} H_2(n)\, e^{-nz} = \big(1 + o(1)\big)\, \frac{4A_1^2}{\sqrt{2\pi}}\, z^{-\frac{3}{2}} \exp\left(\frac{\pi^2}{6z}\right) \tag{11.7}$$

if $z \to 0$ in any angle from the right halfplane. Since $H_2(n)$ is monotonically increasing, we have here

$$H_2(1)\, e^{-z} + \sum_{n=2}^{\infty} \big(H_2(n) - H_2(n-1)\big)\, e^{-nz} = \big(1 + o(1)\big) \frac{4A_1^2}{\sqrt{2\pi}} z^{-\frac{1}{2}} \exp\left(\frac{\pi^2}{6z}\right). \tag{11.8}$$

Since the coefficients are now nonnegative, Ingham's above quoted theorem is applicable; this completes the proof of Lemma III.

12. The proof of our theorem can be completed by considering the expression

$$S = \frac{1}{p(n)} \sum_K \left(\frac{\log O(K)}{\sqrt{n}} - A_0\right)^2 \tag{12.1}$$

(for A_0 see (1.5) and (3.3)). This is

$$\frac{1}{n\,p(n)} H_2(n) - \frac{2A_0}{\sqrt{n}\,p(n)} H_1(n) + A_0^2$$

which is $o(1)$ owing to Lemma II and III. Thus the number $A(n)$ of classes K satisfying

$$\left|\frac{\log O(K)}{\sqrt{n}} - A_0\right| > \delta$$

with a fixed $\delta > 0$ is such that

$$\frac{A(n)}{p(n)}\, \delta^2 = o(1)$$

which proves the theorem.

13. Another proof of our theorem can be based on the following lemma, interesting perhaps also on his own.

Lemma. *For a suitable continuous and monotonically decreasing $f(c)$ almost all partitions of n contain for $n \to \infty$, (δ small positive and fixed)*

$$\big(1 + o(1)\big)\, f(c)\, \{\pi(\sqrt{n}(1+\delta)) - \pi(\sqrt{n})\}$$

p prime summands from the interval

$$c\sqrt{\ } \leq p \leq (c+\delta)\sqrt{n}.$$

Here

$$f(0)=1, \quad \lim_{r\to+\infty} f(r) = 0$$

and $\pi(m)$ as usual stands for the number of primes not exceeding m.

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