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# On a linear diophantine problem of Frobenius

by

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**Introduction.** Given integers $0 < a_1 < \ldots < a_n$ with $\gcd(a_1, \ldots, a_n) = 1$, it is well-known that the equation $N = \sum_{k=1}^{n} x_k a_k$ has a solution in non-negative integers $x_k$ provided $N$ is sufficiently large. Following [9], we let $G(a_1, \ldots, a_n)$ denote the greatest integer $N$ for which the preceding equation has no such solution.

The problem of determining $G(a_1, \ldots, a_n)$, or at least obtaining non-trivial estimates, was first raised by G. Frobenius (cf. [2]) and has been the subject of numerous papers (e.g., cf. [1], [2], [3], [4], [7], [8], [9], [11], [12], [13]). It is known that:

$$G(a_1, a_2) = (a_1-1)(a_2-1)-1 \qquad ([2], [11]);$$

$$G(a_1, \ldots, a_n) \leqslant (a_1-1)(a_n-1)-1 \qquad ([2], [4]);$$

$$G(a_1, \ldots, a_n) \leqslant \sum_{k=1}^{n-1} a_{k+1} d_k / d_{k+1}$$

where $d_k = \gcd(a_1, \ldots, a_k)$ ([2]). The exact value of $G$ is also known for the case in which the $a_k$ form an arithmetic progression ([1], [13]).

In this paper, we obtain the bound

$$G(a_1, \ldots, a_n) \leqslant 2a_{n-1}\left[\frac{a_n}{n}\right] - a_n,$$

which in many cases is superior to previous bounds and which will be seen to be within a constant factor of the best possible bound. We also consider several related extremal problems and obtain an exact solution in the case that $a_n - 2n$ is small compared to $n^{1/2}$.

**A general bound.** As before, we consider integers $0 < a_1 < \ldots < a_n$ with $\gcd(a_1, \ldots, a_n) = 1$.

THEOREM 1.

$$G(a_1, \ldots, a_n) \leqslant 2a_{n-1}\left[\frac{a_n}{n}\right] - a_n. \tag{1}$$

Proof. Let $g$ denote $a_n$, let $m$ denote $\left[\frac{a_n}{n}\right]$ and let $A$ denote the set $\{0, a_1, \ldots, a_{n-1}\}$ of residues modulo $g$. Consider the sum

$$\mathscr{C} = \underbrace{A + \ldots + A}_{m} = \{b_1 + \ldots + b_m \colon b_k \in A\} \pmod{g}.$$

By a strong theorem of Kneser ([10]; cf. also [6], p. 57), there exists a (minimal) divisor $g'$ of $g$ such that

$$\mathscr{C} = \underbrace{A^{(g')} + \ldots + A^{(g')}}_{m} \pmod{g}$$

where

$$A^{(g')} = \{a + rg' \colon 0 \leqslant r < g/g',\ a \in A\} \pmod{g}$$

and such that

$$\frac{|\mathscr{C}|}{g} \geqslant \frac{mn}{g} - \frac{m-1}{g'}. \tag{2}$$

Assume $\mathscr{C}$ does not contain a complete system of residues modulo $g$. Since $\gcd(a_1, \ldots, a_{n-1}, g) = 1$ then $A^{(g')}$ must consist of more than one congruence class mod $g'$. By the theorem of Kneser and the minimality of $g'$, it follows that $\mathscr{C}$ must contain at least $m+1$ distinct residue classes mod $g'$; thus

$$\frac{|\mathscr{C}|}{g} \geqslant \frac{m+1}{g'}. \tag{3}$$

Note that $g \geqslant n$ and $m = [g/n]$ imply

$$m+1 > \frac{1}{2}\left(\frac{m-1}{\frac{mn}{g} - \frac{1}{2}}\right). \tag{4}$$

Suppose now that $|\mathscr{C}| \leqslant \frac{1}{2}g$. By (2) and (4) we have

$$\frac{mn}{g} - \frac{m-1}{g'} \leqslant \frac{1}{2}, \quad \cdot g' \leqslant \frac{m-1}{\frac{mn}{g} - \frac{1}{2}} < 2(m+1).$$

Hence, by (3),

$$\frac{|\mathscr{C}|}{g} \geqslant \frac{m+1}{g'} > \frac{m+1}{2(m+1)} = \frac{1}{2}$$

which is a contradiction.

We may therefore assume $|\mathscr{C}| > \frac{1}{2}g$. But in this case it is easily seen that $\mathscr{C}+\mathscr{C}$ contains a complete residue system mod $g$. It follows that the least possible integer not representable in the form

$$x_1 b_1 + \ldots + x_{2m} b_{2m} + xg$$

with $x_k \geqslant 0$, $x \geqslant 0$, $b_k \in A$, is given by

$$2m \cdot \max_{a \in A}(a) - g = 2a_{n-1}\left[\frac{a_n}{n}\right] - a_n.$$

This proves the theorem.

Note that in the case that $n = 2$ and $a_2$ is odd we have

$$G(a_1, a_2) \leqslant 2a_1\left[\frac{a_2}{2}\right] - a_2 = a_1 a_2 - a_1 - a_2$$

which is best possible.

**An extremal problem.** The question of the estimation of $G$ naturally suggests the following extremal problem. For integers $n$ and $t$, define $g(n, t)$ by

$$g(n, t) = \max_{a_i} G(a_1, \ldots, a_n)$$

where the max is taken over all $a_i$ satisfying

$$(5) \qquad 0 < a_1 < \ldots < a_n \leqslant t, \quad \gcd(a_1, \ldots, a_n) = 1.$$

By Theorem 1 the following result is immediate.

COROLLARY. $g(n, t) < 2t^2/n$.

On the other hand, it is not hard to see that for the set $\{x, 2x, \ldots, (n-1)x, x^*\}$ with $x = [t/(n-1)]$ and $x^* = (n-1)[t/(n-1)]-1$,

$$g(n, t) \geqslant G(x, \ldots, x^*) \geqslant \frac{t^2}{n-1} - 5t \quad \text{for} \quad n \geqslant 2.$$

Thus, $g(n, t)$ is bounded below by essentially $t^2/n$.

Of course, for $n = 2$, the exact value of $g$ is given by $g(2, t) = (t-1)(t-2)-1$. It appears that

$$g(3, t) = \left[\frac{(t-2)^2}{2}\right] - 1,$$

with the sets $\{t/2, t-1, t\}$ or $\{t-2, t-1, t\}$ for $t$ even and $\{(t-1)/2, t-1, t\}$ for $t$ odd achieving this bound. However, this has not yet been established. It follows from the Corollary that $g(n, cn) < 2c^2 n$ and $g(n, n^2) < 2n^3$; again, the truth probably differs from these estimates by a factor of $1/2$ for large $n$.

**Determination of** $g(n, 2n+k)$. The remainder of the paper will be concerned with the determination of $g(n, 2n+k)$ for $n$ large compared to $k$. It follows easily from density considerations that $g(n, 2n+k) = 2n+2k-1$ for $k \leqslant -1$ (cf. [12]). It was shown in [5] that $g(n, 2n) = 2n+1$ and $g(n, 2n+1) = 2n+3$. It was also proved in [5] that for $k$ fixed $g(n, 2n+k) = 2n+h(k)$ for some function $h$ of $k$ provided $n$ is sufficiently large. The exact value of $h(k)$ is given by the next result.

THEOREM 2. *For $k$ fixed, if $n$ is sufficiently large then*

$$g(n, k) = \begin{cases} 2n+2k-1 & \text{for} \quad k \leqslant -1, \\ 2n+1 & \text{for} \quad k = 0, \\ 2n+4k-1 & \text{for} \quad k \geqslant 1 \text{ and } n-k \equiv 1 \pmod 3, \\ 2n+4k+1 & \text{for} \quad k \geqslant 1 \text{ and } n-k \not\equiv 1 \pmod 3). \end{cases}$$

Proof. By previous remarks we may restrict ourselves to $k \geqslant 2$. Assume for a fixed integer $K \geqslant 2$ the theorem holds for all $k < K$. Let $A = \{a_1, \ldots, a_n\}$ be a set satisfying (5) with $k = K$ and $n$ large (to be specified later). We first establish

$$g(n, k) \leqslant \begin{cases} 2n+4K-1 & \text{if} \quad n-K \equiv 1 \pmod 3, \\ 2n+4K+1 & \text{if} \quad n-K \not\equiv 1 \pmod 3). \end{cases} \tag{6}$$

Let $S(A)$ denote the set of sums $\{\sum_{i=0}^{n} x_i a_i \colon x_i \geqslant 0\}$ we are considering and let $G(A)$ abbreviate $G(a_1, \ldots, a_n)$. Note that if there exists an $x$, $1 \leqslant x \leqslant 2n+K$, with $x \in S(A)$, $x \notin A$, then the set $A' = A \cup \{x\}$ satisfies

$$0 < a_1' < \ldots < a_{n+1}' = 2n+K = 2(n+1)+K-2.$$

By the induction hypothesis

$$G(A) = G(A') \leqslant 2(n+1)+4(K-2)+1 = 2n+4K-5 < 2n+4K-1$$

so that (6) certainly holds in this case. Hence, we may assume $A$ and $S(A)$ agree below $2n+K$.

Next, suppose $2n+K+1 \in S(A)$. Then for $A' = A \cup \{2n+K+1\}$ we have

$$0 < a_1' < \ldots < a_{n+1}' = 2n+K+1 = 2(n+1)+K-1$$

so that by the induction hypothesis

$$G(A) = G(A') \leqslant 2(n+1)+4(K-1)+1 = 2n+4K-1$$

and (6) holds in this case. Hence, we may assume

$$2n+K+1 \notin S(A).$$

Now, suppose $2n+K+2\in S(A)$, $2n+K+3\in S(A)$. For $A' = A\cup \cup \{2n+K+2, 2n+K+3\}$ we have

$$0 < a_1' < \ldots < a_{n+2}' = 2n+K+3 = 2(n+2)+K-1.$$

By the induction hypothesis

$$G(A) = G(A') \leqslant \begin{cases} 2(n+2)+4(K-1)-1 & \text{if } (n+2)-(K-1) \equiv 1 (\operatorname{mod} 3), \\ 2(n+2)+4(K-1)+1 & \text{if } (n+2)-(K-1) \not\equiv 1 (\operatorname{mod} 3) \end{cases}$$
$$= \begin{cases} 2n+4K-1 & \text{if } n-k \equiv 1 \ (\operatorname{mod} 3), \\ 2n+4K+1 & \text{if } n-k \not\equiv 1 \ (\operatorname{mod} 3), \end{cases}$$

so that (6) holds in this case. Hence we may assume that either

$$2n+K+2\notin S(A) \quad \text{ or } \quad 2n+K+3\notin S(A).$$

There are two cases:

(I) Suppose $a_1 \leqslant 3K$. If at least $3K$ consecutive integers belong to $A$ then by successively adding $a_1$ to these integers, we infer that $G(A) < 2n+K$ and (6) holds in this case. Therefore, we may assume that $A$ does not contain $3K$ consecutive integers.

Since we have assumed $2n+K+1\notin S(A)$ then for all $i$, $1\leqslant i\leqslant 2n+K$, either $i\notin A$ or $2n+K+1-i\notin A$. Thus, for exactly $\left[\frac{K+1}{2}\right]$ values of $j$ we have $j\notin A$ and $n+K+1-j\notin A$. For a given integer $f(K)$, if $n$ is sufficiently large then for some $t\leqslant\left[\frac{K+1}{2}\right]f(K)$, each of the integers $t+i$, $1\leqslant i\leqslant f(K)$, satisfies either

$$t+i\in A \quad \text{ or } \quad 2n+K+1-(t+i)\in A.$$

Consequently, for some $t'$, $t+1\leqslant t'\leqslant t+3K$, we have

$$2n+K-t'+1\in A.$$

There are several possibilities:

(i) Suppose $2n+K-t'\in A$. If $t'+2\in A$ then we would have $2n+K-t'+2$, $2n+K-t'+3\in S(A)$ which contradicts our assumptions on $A$. We may therefore assume

$$2n+K-t'-1\in A.$$

But now consider $t'+3$. If $t'+3\in A$ then as before we find $2n+K-t'+2$, $2n+K-t'+3\in S(A)$ which is a contradiction. Hence, we must have

$$2n+K-t'-2\in A.$$

We can continue this argument to conclude that

$$2n+K-t'-s \in A \quad \text{for} \quad 0 \leqslant s \leqslant 3K-1,$$

provided $f(K) \geqslant 6K$ and $n$ is sufficiently large. But this is a sequence of $3K$ consecutive integers in $A$ and since this contradicts our assumption on $A$, then case (i) is impossible.

(ii) Suppose $2n+K-t' \notin A$. Then we have

$$t'+1 \in A.$$

If we now have $t'+2 \in A$ then as before $2n+K-t'+2$, $2n+K-t'+3 \in S(A)$ which is a contradiction. Therefore, we may assume $t'+2 \notin A$, i.e.,

$$2n+K-t'-1 \in A.$$

Now, by using the same arguments as in (i) we can argue that $t'+3$, $2n+K-t'-3, \ldots, t'+2r+1$, $2n+K-t'-2r-1 \in A$ for $2r < f(K)-3K$ if $n$ is sufficiently large. In particular we have

$$t'+2j+1 \in A, \quad 0 \leqslant j < \tfrac{1}{2}\big(f(K)-3K\big)$$

where $t' \leqslant \left[\frac{K+1}{2}\right] f(K)+3K$. Since $a_1 \leqslant 3K$ then by successively adding $2a_1$ to the integers $t'+2j+1$, we see that all integers $x$ of the form $x = t'+$ $+2s+1$, $s \geqslant 0$, belong to $S(A)$ provided

$$6K \leqslant f(K)-3K.$$

Of course if $t' \equiv 0 \pmod 2$, then by adding $t'+1 \in A$ to the integers $t'+2s+1$, $s \geqslant 0$, we see that all integers $\geqslant 2\left[\frac{K+1}{2}\right] f(K)+6K+2$ belong to $S(A)$. For $n$ sufficiently large, this certainly implies (6). We may therefore assume

$$t' \equiv 1 \pmod 2$$

and consequently all even integers $\geqslant t'+1$ belong to $S(A)$. In fact, is it clear that if $x \in A$ is an odd integer and $x \leqslant 2n+K-(t'+1)$ then all odd integers $\geqslant 2n+K$ (and hence all integers $\geqslant 2n+K$) belong to $S(A)$. Thus, we may assume that

$$x \in A, \quad x \text{ odd } \Rightarrow x > 2n - \left[\frac{K+1}{2}\right] f(K)-2K.$$

Further, if $K$ is odd then $2n+K+1$ is even and therefore belongs to $S(A)$ for $n$ sufficiently large. This contradicts our assumption on $A$ and we may assume $K$ is even.

Now, let $u$ be the largest integer such that $2n+K-2u+1 \in A$. Since $K$ is even it follows that

$$u < \frac{1}{2}\left(\left[\frac{K+1}{2}\right] f(K) + 3K + 1\right).$$

Consider the $K+1$ integers $2u+2j$, $1 \leqslant j \leqslant K+1$. By the definition of $u$ none of the integers $2n+K-(2u+2j)+1$ belongs to $A$. Since there are at most $\left[\frac{K+1}{2}\right] = \frac{K}{2}$ of these integers for which both $2u+2j \notin A$ and $2n+K-(2u+2j)+1 \notin A$ then we see that at least $K+1-\frac{K}{2} = \frac{K}{2}+1$ of them belong to $A$, say,

$$2u+2j_1, \ldots, 2u+2j_t \in A, \quad t \geqslant K/2+1.$$

Forming the sums

$$(2n+K-2u+1)+(2u+2j_i), \quad i = 1, 2, \ldots, t,$$

we obtain at least $K/2+1$ sums $2n+K+2j_i+1$ which are $\geqslant 2n+K+3$ and $\leqslant 2n+3K+3$ and which belong to $S(A)$. But all the even integers $2n+K+2r$, $1 \leqslant r \leqslant K+1$, also belong to $S(A)$. Hence, $S(A)$ contains at least $n+(K/2+1)+K+1$ integers which are less than or equal to $2n+3K+3$ and we can find a subset $A' \subseteq S(A)$ with

$$0 < a_1' < \ldots < a'_{n+3K/2+2} = 2n+3K+3-d,$$

for some integer $d \geqslant 0$. Since

$$(2n+3K+3-d)-(2+3K/2+2) \leqslant -1$$

then by the induction hypothesis we conclude that all integers $\geqslant 2n+ +3K+3-d$ belong to $S(A)$. If $d \geqslant 1$ then in fact all integers $\geqslant 2n+3K+2$ belong to $S(A)$; if $d = 0$ then since $2n+3K+2$ is even then we still have all integers $\geqslant 2n+3K+2 \in S(A)$. Thus,

$$G(A) \leqslant 2n+3K+1.$$

But for $K \geqslant 2$, $4K-1 \geqslant 3K+1$ so that

$$G(A) \leqslant 2n+4K-1$$

and (6) holds in this case. This concludes case (I).

(II) Suppose $a_1 > 3K$. There are two cases:

(i) Suppose $a_1 > n + \left[\frac{K+1}{2}\right]$. Thus, exactly $\left[\frac{K+1}{2}\right]$ of the integers which are $> n+\left[\frac{K+1}{2}\right]$ and $< 2n+K$ are missing from $A$. This implies that for some $i$, $1 \leqslant i \leqslant \left[\frac{K+1}{2}\right]+1$, both $n+2\left[\frac{K+1}{2}\right]+1+i \epsilon A$ and $n+2\left[\frac{K+1}{2}\right]+2-i \epsilon A$, i.e., $2n+4\left[\frac{K+1}{2}\right]+3 \epsilon S(A)$. Of course, the same argument can be repeated for $2n+4\left[\frac{K+1}{2}\right]+4$, etc., so that for $n$ sufficiently large, $2n+4\left[\frac{K+1}{2}\right]+j+2 \epsilon S(A)$ for $1 \leqslant j \leqslant 4\left[\frac{K+1}{2}\right]+3$. Hence $S(A)$ contains a subset $A'$ with

$$0 < a_1' < \ldots < a'_{n+4\left[\frac{K+1}{2}\right]+3} = 2n+8\left[\frac{K+1}{2}\right]+5-d$$

for some $d \geqslant 0$. Since

$$2\left(n+4\left[\frac{K+1}{2}\right]+3\right) > 2n+8\left[\frac{K+1}{2}\right]+5-d$$

then by the induction hypothesis all integers $> 2n+8\left[\frac{K+1}{2}\right]+5$ belong to $S(A)$. But since $2n+4\left[\frac{K+1}{2}\right]+j+2 \epsilon S(A)$ for $1 \leqslant j \leqslant 4\left[\frac{K+1}{2}\right]+3$ then all integers $> 2n+4\left[\frac{K+1}{2}\right]+2$ belong to $S(A)$. However, $4\left[\frac{K+1}{2}\right]+2 < 4K-1$ for $K \geqslant 2$ so that (6) holds in this case.

(ii) Suppose $a_1 \leqslant n+\left[\frac{K+1}{2}\right]$. Consider the $3K-1$ integers $2n+K-a_1+i+1$, $1 \leqslant i \leqslant 3K-1$. Since $a_1$ is the least element of $A$ then at least $3K-1-\left[\frac{K+1}{2}\right]$ of these integers must belong to $A$. Adding $a_1$ to each of them gives at least $3K-1-\left[\frac{K+1}{2}\right]$ integers in $S(A)$ which are $> 2n+K$ and $\leqslant 2n+4K$. Thus, $S(A)$ contains a subset $A'$ with

$$0 < a_1' < \ldots < a'_{n+3K-1-\left[\frac{K+1}{2}\right]} = 2n+4K-d$$

for some $d \geqslant 0$.

For $K \geqslant 4$,

$$2\left(n+3K-1-\left[\frac{K+1}{2}\right]\right) > 2n+4K-d$$

so that by the induction hypothesis

$$G(A) \leqslant G(A') \leqslant 2n+4K-1$$

and (6) holds. Hence, we may assume $K \leqslant 3$. There are two cases.

Suppose $K = 2$. If $2n - a_1 + j \in A$, $4 \leqslant j \leqslant 6$, then $2n + j \in S(A)$, $4 \leqslant j \leqslant 6$. Thus $S(A)$ contains a subset $A'$ with

$$0 < a_1' < \ldots < a_{n+3}' = 2n+6$$

and by the induction hypothesis

$$G(A) \leqslant G(A') \leqslant 2n+7$$

so that (6) holds in this case.

If at least one of $2n - a_1 + j$, $4 \leqslant j \leqslant 6$, is missing from $A$, then in fact, exactly one of $2n - a_1 + j$, $4 \leqslant j \leqslant 6$, is missing from $A$, and all of $2n - a_1 + j \in A$, $1 \leqslant j \leqslant 9$. Hence, $2n + j \in S(A)$, $7 \leqslant j \leqslant 9$, and $S(A)$ contains a subset $A'$ with

$$0 < a_1' < \ldots < a_{n+5}' \leqslant 2n+9.$$

By the induction hypothesis

$$G(A') \leqslant 2n+8$$

and since $2n+7$, $2n+8 \in S(A)$ then

$$G(A) \leqslant 2n+6$$

which satisfies (6) in this case.

The case $K = 3$ is similar and will be omitted. It can be checked that the condition that $n$ be sufficiently large in the preceding arguments is satisfied, for example, by taking $n > 20K^2$.

This concludes case (II) and (6) is proved.

We next exhibit specific sets $A$ which satisfy (6) with equality for $n$ arbitrarily large. There are three cases.

(i) $n - K \equiv 1 \pmod 3$. Write $n = 3m + K + 1$ and let

$$A = \bigcup_{i=1}^{2m+K} \{3i\} \cup \bigcup_{j=1}^{m+1} \{3m + 3K + 5 - 3j\}.$$

The least element of $S(A)$ which is $\equiv 1 \pmod 3$ is $2(3m+3K+2) = 6m + {} + 6K + 4$ so that

$$2n + 4K - 1 = 6m + 6K + 1 \notin S(A).$$

Therefore $0 < a_1 < \ldots < a_n = 2n + K$ and $G(A) \geqslant 2n + 4K - 1$.

(ii) $n - K \equiv 2 \pmod 3$. Write $n = 3m + K + 2$ and let

$$A = \bigcup_{i=1}^{2m+K+1} \{3i\} \cup \bigcup_{j=1}^{m+1} \{3m + 3K + 7 - 3j\}.$$

(iii) $n - K \equiv 0 \pmod 3$. Write $n = 3m + K$ and let

$$A = \bigcup_{i=1}^{2m+K} \{3i\} \cup \bigcup_{j=1}^{m} \{6m + 3K + 2 - 3j\}.$$

It is easy to see in (ii) and (iii) that $A$ satisfies (5) and $G(A) \geqslant 2n + 4K + 1$.

The examples in (i), (ii) and (iii) together with (6) establish the theorem for $k = K$. This completes the induction step and the theorem is proved.

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Added in proof: The conjecture $g(3, t) = \left[\frac{(t-2)^2}{2}\right] - 1$ has recently been settled in the affirmative by M. Lewin (personal communication).

### References

[1] P. T. Bateman, *Remark on a recent note on linear forms*, Amer. Math. Monthly 65 (1958), pp. 517–518.

[2] Alfred Brauer, *On a problem of partitions*, Amer. J. Math. 64 (1942), pp. 299–312.

[3] — and B. M. Seelbinder, *On a problem of partitions, II*, Amer, J. Math. 76 (1954), pp. 343–346.

[4] — and J. E. Shockley, *On a problem of Frobenius*, J. Reine Angew. Math. 211 (1962), pp. 215–220.

[5] P. Erdös, *Problem P–84*, Can. Math. Bull. 14 (1971), pp. 275–277.

[6] H. Halberstam and K. F. Roth, *Sequences I*, London 1966.

[7] B. R. Heap and M. S. Lynn, *A graph theoretic algorithm for the solution of a linear diophantine equation*, Numerische Math. 6 (1964), pp. 346–354.

[8] — — *On a linear diophantine problem of Frobenius: an improved algorithm*, Numerische Math. 7 (1965), pp. 226–231.

[9] S. M. Johnson, *A linear diophantine problem*, Can. J. Math. 12 (1960), pp. 390–398.

[10] M. Kneser, *Abschätzungen der asymptotischen Dichte von Summenmengen*, Math. Zeitschr. 58 (1953), pp. 459–484.

[11] N. S. Mendelsohn, *A linear diophantine equation with applications to nonnegative matrices*, Ann. N. Y. Acad. Sci. 175 art. 1 (1970), pp. 287–294.

[12] M. Nagata and H. Matsumura, Sûgaku 13 (1961–62), p. 161; Math. Rev. 25 no. 3 # 2386 (1963).

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