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SOME NUMBER THEORETIC RESULTS

(In memory of our good friend Leo Moser)

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The paper first establishes the order of magnitude of maximal sets, S, of residues (mod p) so that the sums of different numbers of elements are distinct.

In the second part irrationalities of Lambert Series of the form $\sum f(n)/a_1 \cdots a_n$ are obtained where $f(n) = d(n)$, $\sigma(n)$ or $\varphi(n)$ and the a_i are integers, $a_i \geqq 2$, which satisfy suitable growth conditions.

This note consists of two rather separate topics. In §1 we generalize a topic from combinatorial number theory to get an order of magnitude for the number of elements in a maximal set of residues (mod p) such that sums of different numbers of elements from this set are distinct. We show that the correct order is $cp^{1/3}$ although we are unable to establish the correct value for the constant c.

Section 2 consists of irrationality results on series of the form $\Sigma f(n)/a_1a_2\cdots a_n$ where $f(n)$ is one of the number theoretic functions $d(n)$, $\sigma(n)$ or $\varphi(n)$ and a_n are integers $\geqq 2$. For $f(n) = d(n)$ it suffices that the a_n are monotonic while for $\sigma(n)$ and $\varphi(n)$ we needed additional conditions on their rates of growth.

1. Maximal sets in a cyclic group of prime order for which subsets of different orders have different sums. In an earlier paper [4] one of us has given a partial answer to the question:

What is the maximal number $n = f(x)$ of integers $a_1, \cdots, a_n$ so that $0 < a_1 < a_2 < \cdots < a_n \leqq x$ and so that

$$a_{i_1} + \cdots + a_{i_s} = a_{j_1} + \cdots + a_{j_t} \text{ for some } \begin{array}{l} 1 \leqq i_1 < \cdots < i_s \leqq n \\ 1 \leqq j_1 < \cdots < j_t \leqq n \end{array}$$

implies $s = t$? it is conjectured that the maximal set is obtained (loosely speaking) by taking the top $2\sqrt{x}$ integers of the interval $(1, x)$. We were indeed able to prove that $f(x) < c\sqrt{x}$ for suitable c (for example $4/\sqrt{3}$) by using the fact that a set of n positive integers has a minimal set of distinct sums of t-tuples $(1 \leqq t \leqq n)$ if it is in arithmetic progression.

It is natural to pose the analogous question for elements of cyclic groups of prime order, as was done at the Number Theory Symposium in Stony Brook [5]. Here again we may conjecture that a maximal set of residues (mod p) is attained by taking a set of consecutive residues, this time not at the upper end but near $p^{2/3}$.

CONJECTURE 1.1. Let $f(p)$ be the maximal cardinality of a set of residues mod p so that sums of different numbers of residues in this set are different, then $f(p) = (4p)^{1/3} + o(p^{1/3})$ where the maximum is attained, for example, by taking consecutive residues in an interval of length $(4p)^{1/3} + o(p^{1/3})$ containing the residue $[(p/2)^{1/3}]$.

It is easy to see that we can indeed get a set of about $(4p)^{1/3}$ residues by taking the residues in the interval $([(p/2)^{2/3} - (4p)^{1/3}], [(p/2)^{2/3}])$. Here sums of distinct numbers of elements are distinct integers, and since all sums are $< p$ it follows that they are distinct residues.

The observation which let to the upper bound in [4] is much less obvious (mod p):

CONJECTURE 1.2. A set $A = \{a_1, a_2, \cdots, a_k\}$ of residues (mod p) has a minimal number of distinct sums of subsets of t elements if A is in arithmetic progression.

Conjecture 1.2 would give us a simple upper bound for $f(p)$:

COROLLARY 1.3. *If Conjecture 1.2 holds then*

$$f(p) < (6p)^{1/3} + o(p^{1/3}) .$$

Proof. The sums of t elements from the set of residues

$$\{1, 2, \cdots, k-1, k\}$$

fill the interval $(\binom{t+1}{2}, tk - \binom{t}{2})$ that is to say there are $tk - t^2 + O(t)$ such sums. Since for different t we get different sums we must have

$$p \geqq \sum_{t=1}^{k} (tk - t^2 + O(t)) = \frac{k^3}{6} + O(k^2)$$

$$\text{and hence } k < (6p)^{1/3} + o(p^{1/3}) .$$

Using methods employed by Erdös and Heilbronn [2] we can show that $f(p) = O(p^{1/3})$. We use the following lemma from [2].

LEMMA 1.4. *Let* $1 < m \leqq l < p/2$ *and let* $B = \{b_1, \cdots, b_l\}$, $A = \{a_1, \cdots, a_m\}$ *be sets of residues* (mod p). *Then there exists an* $a_i \in A$ *such that the number of solutions of* $a_i = b_j - b_k$; $b_j, b_k \in B$ *is less than* $l - m/6$.

We now can get a lower bound for the number of distinct sums of t elements from a set of residues.

LEMMA 1.5. *Let* $A = \{a_1, \cdots, a_k\}$ *be a set of residues* (mod p)

and let $A_t = \{a_{i_1} + \cdots + a_{i_t} | 1 \leqq i_1 < \cdots < i_t \leqq k\}$ *then for* $1 \leqq t \leqq k/4$ *we have*

$$(1.6) \qquad |A_t| \geqq l + \frac{(t-1)m}{6} - \frac{t(t-1)}{6}$$

where

$$l = \left[\frac{k+1}{2}\right], m = \left[\frac{k}{2}\right].$$

Proof. We divide the set A into two disjoint sets

$$A = \{a_1, a_2, \cdots, a_l\}, B = \{b_1, b_2, \cdots, b_m\}$$

and prove the inequality (1.6) for the subset of A_t consisting of the sums

$$A_t{}^* = \{a_i + b_{2-\varepsilon_1} + b_{4-\varepsilon_2} + \cdots + b_{2t-2-\varepsilon_{t-1}} | \varepsilon_j = 0 \text{ or } 1\},$$

where the b_i are a suitable ordering of the elements of B.

The inequality holds for $t = 1$ since

$$A_t{}^* = \{a_i\} = A \text{ and } |A| = l.$$

Now assume that (1.6) holds for $A_t{}^*$ with $t \leqq (m/2) - 1$. Then the set $A_t{}^* + b_{2t} \subset A^*{}_{t+1}$ and according to Lemma 1.3 there exists a $b_j \in \{b_{2t+1}, b_{2t+1}, \cdots, b_m\}$, say $b_j = b_{2t+1}$ so that the equation

$$b_{2t+1} - b_{2t} = a_i^* - a_j^*, \;\; a_i^*, a_j^* \in A_t^*$$

has no more than $|A_t^*| - \frac{1}{6}(m-2t)$ solutions. Hence the set

$$((b_{2t+1} - b_{2t}) + (A_t^* + b_{2t})) \cap (A_t^* + b_{2t})$$

contains no more than $A_t^* - \frac{1}{6}(m-2t)$ elements and

$$\begin{aligned} |A_{t+1}^*| &= |(A_t^* + b_{t+1}) \cup (A_t^* + b_t)| \\ &\geqq |A_t^*| + \tfrac{1}{6}(m-2t) \\ &\geqq l + \frac{(t-1)m}{6} - \frac{t(t-1)}{6} + \frac{1}{6}m - \frac{t}{3} \\ &= l + \frac{tm}{6} - \frac{(t+1)t}{6}. \end{aligned}$$

This completes the proof.

THEOREM 1.7. *The maximal number* $f(p)$ *of a set* A *of residues* (mod p) *so that sums of different numbers of distinct elements of* A *are distinct satisfies*

$$(1.8) \qquad (4p)^{1/3} + o(p^{1/3}) < f(p) < (288p)^{1/3} + o(p^{1/3}).$$

Proof. According to Lemma 1.5 there are at least

$$k/2 + k(t-1)/12 - t^2/6 + O(t)$$

distinct sums of t elements (and hence, by symmetry, sums of $k-t$ elements) for $t < [k/4]$ out of a set A with k elements. Thus if A has the desired property we must have

$$\begin{aligned} p &\geqq 2\sum_{t=1}^{k/4}(k/2 + k(t-1)/12 - t^2/6) + O(k^2) \\ &= 2k^3\Big(\frac{1}{384} - \frac{1}{3}\,\frac{1}{384}\Big) + O(k^2) = k^3/288 + O(k^2)\,. \end{aligned}$$

Thus

$$f(p) < (288\,p)^{1/3} + o(p^{1/3})\,.$$

The lower bound for $f(p)$ was established above.

2. **On some irrational series.** One of us [1] proved that the series $\sum_{n=1}^{\infty} d(n)t^{-n}$ is irrational for every integer $t, |t| > 1$. In this section we generalize this result to series of the form

$$\xi = \sum_{n=1}^{\infty} \frac{d(n)}{a_1 a_2 \cdots a_n} \tag{2.1}$$

where the a_n are positive integers with $2 \leqq a_1 \leqq a_2 \leqq \cdots$. It is clear that we need some restriction, such as monotonicity, on the a_n since the choice $a_n = d(n) + 1$ would lead to $\xi = 1$.

We divide the proof into two cases depending on the rate of increase of a_n. The first case is very similar to [1].

LEMMA 2.2. *The series (2.1) is irrational if there exists a $\delta > 0$ so that the inequality $a_n < (\log n)^{1-\delta}$ holds for infinitely many values of n.*

Proof. Let n be a large integer so that $a_n < (\log n)^{1-\delta}$. Then by the monotonicity of a_i there exists an interval I of length $n/\log n$ in $(1, n)$ so that for all integers $i \in I$ we have $a_i = t$ where t is a fixed integer, $t \leqq (\log n)^{1-\delta}$.

Now put $k = [(\log n)^{\delta/10}]$ and let $p_1, p_2, \cdots$ be the consecutive primes greater than $(\log n)^2$. Let

$$A = \Big(\prod_{1 \leqq i \leqq k(k+1)/2} p_i\Big)^t$$

then

$$\begin{aligned} A &< (2(\log n)^2)^{t\,k(k+1)/2} < e^{(\log n)^{1-\delta}(\log n)^{\delta/4}} \\ &< e^{(\log n)^{1-\delta/2}}\,. \end{aligned} \tag{2.3}$$

By the Chinese remainder theorem the congruences

$$(2.4)\qquad \begin{aligned} x &\equiv p_1^{t-1} (\bmod p_1^t)\\ x+1 &\equiv (p_2p_3)^{t-1} (\bmod (p_2p_3)^t)\\ &\vdots\\ x+k-1 &\equiv (p_u p_{u+1}\cdots p_{u+k-1})^{t-1} (\bmod (p_u p_{u+1}\cdots p_{u+k-1})^t)\end{aligned}$$

where $u = 1 + k(k-1)/2$, have solutions determined (mod A). The interval I contains at least $[n/(A \log n)]$ solutions of (2.4).

Now assume that $\xi = a/b$ and choose $x \in I$ to be a solution of (2.4) so that $(x, x+k) \subset I$. Then

$$(2.5)\qquad \begin{aligned} b\,a_1\cdots a_{x-1}\xi &= \text{integer} + b\sum_{l=0}^{k-1} \frac{d(x+l)}{t^{l+1}}\\ &\quad + b\sum_{s=0}^{\infty} \frac{d(x+k+s)}{t^k a_{x+k}\cdots a_{x+k+s}}\,. \end{aligned}$$

But (2.4) implies that $d(x+l) \equiv 0 \,(\bmod t^{l+1})$ for $l = 0, 1, \cdots, k-1$. Thus (2.5) implies that

$$(2.6)\qquad b\,a_1\cdots a_{x-1}\xi = \text{integer} + \frac{b}{t^k}\sum_{s=0}^{\infty} \frac{d(x+k+s)}{a_{x+k}\cdots a_{x+k+s}}\,.$$

We now wish to show that for suitable choice of x the sum on the right side of (2.6) is less than 1 and hence $b\xi$ cannot be an integer. We first consider the sum

$$(2.7)\qquad \begin{aligned} &\frac{b}{t^k}\sum_{s>10\log n} \frac{d(x+k+s)}{a_{x+k}\cdots a_{x+k+s}}\\ &< \frac{b}{t^k}\sum_{s>10\log n} \frac{x+k+s}{t^{s+1}} < b(x+k)\sum_{s>10\log n} \frac{s}{t^s}\\ &< \frac{2bn}{n^2} < \frac{1}{2} \text{ for large } n\,. \end{aligned}$$

Next we wish to show that it is possible to choose x so that

$$(2.8)\qquad d(x+k+s) < 2^{k/4} \text{ for } 0 \leq s < 10 \log n\,.$$

We first observe that

$$(2.9)\qquad (x+k+s, A) = 1 \text{ for all } 0 \leq s < 10 \log n$$

since otherwise

$$(2.10)\qquad x+k+s \equiv 0 \,(\bmod p_j) \text{ for some } 1 \leq j \leq k(k+1)/2$$

and

$$(2.11)\qquad x+i \equiv 0 \,(\bmod p_j) \text{ for some } 0 \leq i < k\,.$$

But

$$0 < k + s - i < 11 \log n < (\log n)^2 < p_i$$

so that (2.10) and (2.11) are incompatible.

Let $x = x_0, x_0 + A, \cdots, x_0 + zA$ be the solutions of (2.4) for which $(x, x + k) \subset I$. From (2.9) we get

$$\begin{aligned}(2.12)\qquad \sum_{y=0}^{z} d(x_0 + k + s + yA) &< 2\sum_{l=1}^{\sqrt{n}}\left(\frac{n}{Al} + 1\right)\\ &< c\,\frac{n \log n}{A}\,.\end{aligned}$$

Thus the number of y's for which $d(x_0 + k + s + yA) > 2^{k/4}$ is less than $c\,n\ \log n/(A.2^{k/4})$, and the number of y's so that for some $0 \leqq s < 10 \log n$ we have $d(x_0 + k + s + yA) > 2^{k/4}$ is less than

$$10c\, n \log^2 n/(A.2^{k/4}) < 1/2\, n/(A \log n) < z\,.$$

It is therefore possible to choose $x = x_0 + yA \in I$ so that (2.8) holds. For such a choice we get

$$\begin{aligned}(2.13)\qquad \frac{b}{t^k}\sum_{s=0}^{10\log n}\frac{d(x + k + s)}{a_{x+k}\cdots a_{x+k+s}} &< \frac{b}{t^k}\,2^{k/4}\sum_{s=0}^{\infty}\frac{1}{t^s}\\ &< b.2^{-3k/4} < \frac{1}{2}\,.\end{aligned}$$

Combining (2.7) and (2.13) we see that ξ is irrational.

LEMMA 2.14. *If there exists a positive constant c so that $|a_n| > c(\log n)^{3/4}$ for all n then the series (2.1) is irrational.*

Note that in this lemma we need not assume the monotonicity of a_n (or even that they are positive, however for simplicity we give the proof for positive a_n only).

Proof. We use two results. The Dirichlet divisor theorem

$$(2.15)\qquad \sum_{n=1}^{N} d(n) \sim N \log N$$

and the average order of $d(n)$, [3]

$$(2.16)\qquad d(n) < (\log n)^{\log 2+\epsilon} \text{ for almost all } n\,.$$

From (2.15) we get the following.

LEMMA 2.17. *Given constants $b, c > 0$, then for almost all integers x*

$$d(x+y) < b^{-1}(2c)^{-y}(\log x)^{3y/4};\ y = 3, 4, \cdots \tag{2.18}$$

Proof. If we choose x large enough so that $\log x > (2bce)^{4/3}$ then the right side of (2.18) is greater than e^y which exceeds $x + y$, and hence $d(x + y)$, whenever $y > 2 \log x$. Thus, if (2.18) fails to hold for sufficiently large x then it must fail to hold for some y with $3 \leq y \leq 2 \log x$.

Now if there are $c_1 N$ integers x below N so that (2.18) fails to hold then we have more than $c_2 N$ integers x with $\sqrt{N} \leq x \leq N - 2 \log N$ and

$$\begin{aligned} d(x+y) &> b^{-1}(2c)^{-y}(\log x)^{3y/4} \geqq b^{-1}(2c)^{-y}(\tfrac{1}{2}\log N)^{3y/4} \\ &\geqq b^{-1}(4c)^{-3}(\log N)^{9/4} = c_3(\log N)^{9/4}\,. \end{aligned} \tag{2.19}$$

Thus

$$\begin{aligned} \sum_{n=1}^{N} d(n) &\geqq c_2 N \cdot \frac{1}{2 \log N} c_3 (\log N)^{9/4} \\ &= c_4 N(\log N)^{5/4} \end{aligned}$$

which contradicts (2.15) for large N.

Combining Lemma 2.17 with (2.16) we find that there exists an infinite set S of integers x so that

$$d(x+1) < \frac{b^{-1}c}{2}(\log x)^{3/4},\ d(x+2) < \frac{b^{-1}c^2}{4}(\log x)^{6/4} \tag{2.21}$$

and (2.18) both hold.

Now assume that $\xi = a/b$ is a rational value of (2.1) and choose $n \in S$. Then

$$a_1 \cdots a_n b\xi = \text{integer} + b \sum_{y=1}^{\infty} \frac{d(n+y)}{a_{n+1} \cdots a_{n+y}} \tag{2.22}$$

where

$$0 < \sum_{y=1}^{\infty} \frac{d(n+y)}{a_{n+1} \cdots a_{n+y}} < \sum_{y=1}^{\infty} \frac{(2c)^{-y}(\log n)^{3y/4}}{(c(\log n)^{3/4})^y} = 1\,,$$

in contradiction to the fact that the left side of (2.22) is an integer.

Summing up we have

THEOREM 2.23. *The series* (2.1) *is irrational whenever*

$$2 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n \leqq \cdots\,.$$

With considerable additional effort one can weaken the monotonicity condition on the a_n to $a_m/a_n \geqq c > 0$ for all $m > n$.

We have not been able to prove the following

Conjecture 2.24. The series (2.1) is irrational whenever $a_n \to \infty$.

If we consider series of the form

(2.25) $$\sum_{n=1}^{\infty} \frac{\varphi(n)}{a_1 \cdots a_n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{\sigma(n)}{a_1 \cdots a_n}$$

then we cannot make conjectures analogous to 2.24 since the choice $a_n = \varphi(n) + 1$ or $\sigma(n) + 1$ would make these series converge to 1. It is reasonable to conjecture that the series (2.25) must be irrational if the a_n increase monotonically, however we can prove this only under more restrictive conditions.

THEOREM 2.26. *If $\{a_n\}$ is a monotonic sequence of integers with $a_n \geqq n^{11/12}$ for all large n then the series in (2.25) are irrational.*

For the proof we need the following simple lemmas.

LEMMA 2.27. *Let $\{a_n\}$ be a sequence of positive integers with $a_n \geqq 2$ and $\{b_n\}$ a sequence of positive integers so that $b_{n+1} = o(a_n a_{n+1})$. If*

(2.28) $$\xi = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$$

is rational then $a_n = O(b_n)$.

Proof. Assume $\xi = a/b$ and choose N so that for all $n > N$ we have $bb_n < a_{n-1}a_n/4$. If there existed an $n > N$ so that $a_n > 2bb_n$ then we would have

$$ba_1 \cdots a_{n-1}\xi = aa_1 \cdots a_{n-1} = \text{integer} + \sum_{k=0}^{\infty} \frac{bb_{n+k}}{a_n \cdots a_{n+k}}$$

but

$$0 < \sum_{k=0}^{\infty} \frac{bb_{n+k}}{a_n \cdots a_{n+k}} = \frac{bb_n}{a_n} + \sum_{k=1}^{\infty} \frac{bb_{n+k}}{a_{n+k-1} \cdots a_{n+k}} \cdot \frac{1}{a_n \cdots a_{n+k-2}}$$
$$< \tfrac{1}{2} + \tfrac{1}{4}\sum_{l=0}^{\infty} (\tfrac{1}{2})^l = 1 ,$$

a contradiction. Thus $a_n \leqq 2bb_n$ for all large n.

LEMMA 2.29. *If the series (2.28) is rational, say $\xi = a/b$, and $b_{n+1} = o(a_n a_{n+1})$, then there exists a sequence of positive integers $\{c_n\}$ so that for all large n we have*

(2.30) $$bb_n = c_n a_n - c_{n+1}, \quad 0 < c_{n+1} < a_n, \text{ and } c_{n+1} = o(a_n) .$$

Conversely, if these conditions hold then the series (2.28) is rational.

Proof. Choose N so that for all $n > N$ we have $bb_n < a_n a_{n+1}/4$. Now for $n \geqq N$ choose c_n, c_{n+1} so that

$$bb_n = c_n a_n - c_{n+1}\,, \qquad c_n > 0$$
$$0 < c_{n+1} < a_n$$

and c'_{n+1}, c'_{n+2}

$$bb_{n+1} = c'_{n+1} a_{n+1} - c'_{n+2}\,, \qquad c'_{n+1} > 0$$
$$0 < c'_{n+2} < a_{n+1}\,.$$

Then

$$\begin{aligned} ba_1 \cdots a_{n-1}\xi &= aa_1 \cdots a_{n-1} \\ &= \text{integer} + \frac{bb_n}{a_n} + \frac{bb_{n+1}}{a_n a_{n+1}} + \sum_{k=2}^{\infty} \frac{bb_{n+k}}{a_n \cdots a_{n+k}} \\ &= \text{integer} - \frac{c_{n+1}}{a_n} + \frac{c'_{n+1}}{a_n} - \frac{c'_{n+2}}{a_n a_{n+1}} \\ &\quad + \frac{1}{a_n}\sum_{k=2}^{\infty} \frac{bb_{n+k}}{a_{n+1} \cdots a_{n+k}} \\ &= \text{integer} - \frac{c_{n+1}}{a_n} + \frac{c'_{n+1}}{a_n} - \frac{c'_{n+2}}{a_n a_{n+1}} + \frac{\theta}{a_n}\,, \\ &\qquad 0 < \theta < \tfrac{1}{2}\,. \end{aligned} \tag{2.31}$$

Thus

$$\frac{1}{a_n}\left(-c_{n+1} + c'_{n+1} - \frac{c'_{n+2}}{a_{n+1}} + \theta\right) = \text{integer}$$

and since $0 < c_{n+1} < a_n$, $0 < c'_{n+1} \leqq [a_n/4] + 1$ $0 < c'_{n+2}/a_{n+1} < 1$, $0 < \theta < \frac{1}{2}$, this is possible only if $c_{n+1} = c'_{n+1}$.

Now choose N so large that $bb_{n+1} < \varepsilon a_n a_{n+1}$ for all $n > N$, then from (2.31) we have

$$\begin{aligned} \text{integer} &= -\frac{c_{n+1}}{a_n} + \sum_{k=1}^{\infty} \frac{bb_{n+k}}{a_n a_{n+1} \cdots a_{n+k}} < -\frac{c_{n+1}}{a_n} + \varepsilon \sum_{k=1}^{\infty} \frac{1}{a_n \cdots a_{n+k-2}} \\ &\leqq -\frac{c_{n+1}}{a_n} + 2\varepsilon\,. \end{aligned}$$

Thus $c_{n+1} < 2\varepsilon a_n$ for all $n > N$.

If condition (2.30) holds for all $n \geqq N$ then

$$\begin{aligned} \sum_{n=N}^{\infty} \frac{bb_n}{a_1 \cdots a_n} &= \sum_{n=N}^{\infty} \frac{c_n a_n - c_{n+1}}{a_1 \cdots a_n} \\ &= \frac{c_N}{a_1 \cdots a_{N-1}} - \sum_{n=N}^{\infty} c_{n+1}\left(\frac{1}{a_1 \cdots a_n} - \frac{a_{n+1}}{a_1 \cdots a_{n+1}}\right) \\ &= \frac{c_N}{a_1 \cdots a_{N-1}} \end{aligned}$$

is clearly rational.

Finally we need a fact from sieve theory. We are grateful to R. Miech for supplying the correct constants.

LEMMA 2.32. *Given an integer a and $\varepsilon > 0$ then for large y the number of integers m satisfying*

$$m \not\equiv 0,\ m \not\equiv a \pmod{p}$$

for all primes p, with $2 < p < y^{1/3}$ exceeds $y^{1-\varepsilon}$.

Proof of Theorem 2.26. Let $f(n)$ stand for either $\sigma(n)$ or $\varphi(n)$ and assume that

$$\sum_{n=1}^{\infty} \frac{f(n)}{a_1 \cdots a_n} = \frac{a}{b}\ .$$

Since $a_n > n^{11/12}$ for large n the hypothesis of Lemma 2.29 is satisfied and we get

(2.33) $$bf(n) = c_n a_n - c_{n+1} \text{ for large } n\ .$$

Since $f(n) = o(n^{1+\varepsilon})$ for all $\varepsilon > 0$ we get

(2.34) $$c_n < n^{1/12+\varepsilon} \text{ for large } n\ .$$

From Lemma 2.28 we get

(2.35) $$a_n = O(f(n)) = O(n^{1+\varepsilon})$$

and hence the number of integers $n \leq x$ for which

$$\frac{a_{n+1}}{a_n} > 1 + x^{-1/2}$$

is $O(x^{3/4})$, since otherwise we would have

$$a_x = \prod_{n<x} \frac{a_{n+1}}{a_n} > (1 + x^{-1/2})^{x^{3/4}} > x^2$$

for large x, in contradiction to (2.35). From now on we restrict our attention to integers n for which

(2.36) $$\frac{a_{n+1}}{a_n} < 1 + n^{-1/2}\ .$$

For such integers we get from (2.33) and (2.35) that

$$\frac{f(n+1)}{f(n)} = \frac{c_{n+1}\,a_{n+1}}{c_n\,a_n}\left(1 - \frac{c_{n+2}}{c_{n+1}\,a_{n+1}}\right)\Big/\left(1 - \frac{c_{n+1}}{c_n\,a_n}\right)$$

$$= \frac{c_{n+1}}{c_n}(1 + O(n^{-1/2}))\,(1 + O(n^{-3/4+\varepsilon})) \tag{2.37}$$

$$= \frac{c_{n+1}}{c_n} + O(n^{-1/2+\varepsilon})$$

Now consider a prime q, $\frac{1}{2}x^{1/11} \leqq q \leqq x^{1/11}$, then according to Lemma 2.32 there exist more than $y^{1-\varepsilon}$ integers $m \leqq y = x^{10/11}$ so that

$$m \not\equiv 0,\ m \not\equiv -2q \pmod{p} \tag{2.38}$$

for all primes p with $2 < p < y^{1/5}$. We may even assume that m is odd. The number of integers $n = 2qm$ where m satisfies (2.38) exceeds $x^{10/11-\varepsilon} > x^{3/4}$ and hence we can pick such an n that satisfies (2.37) with $x/2 \leqq n \leqq x$.

Now

$$f(n) = f(2q)f(m)$$

where

$$\frac{f(2q)}{2q} = \begin{cases} \dfrac{3(q+1)}{2q} & \text{if } f = \sigma \\[2mm] \dfrac{q-1}{2q} & \text{if } f = \varphi \end{cases}$$

in either case

$$f(2q) = A/q, \quad A \text{ an integer not divisible by } q. \tag{2.39}$$

Since m has at most 5 prime factors all exceeding $y^{1/5}$ we have

$$(1 - y^{-1/5})^5 < \frac{f(m)}{m} < (1 + y^{-1/5})^5$$

$$f(m) = m(1 + O(y^{-1/5})) = m\,(1 + O(x^{-2/11}))\,. \tag{2.40}$$

By the same reasoning we get

$$f(n+1) = n(1 + O(x^{-2/11}))\,. \tag{2.41}$$

Substituting (2.39), (2.40) and (2.41) in (2.37) we get

$$\frac{f(n+1)}{f(n)} = \frac{A}{q}\,(1 + O(x^{-2/11})) = \frac{c_{n+1}}{c_n} + O(x^{-1/2+\varepsilon})\,. \tag{2.42}$$

But since $q > x^{1/12}$ and $c_n < x^{1/12}$ we get

$$\frac{1}{qc_n} \leqq \left|\frac{A}{q} - \frac{c_{n+1}}{c_n}\right| < x^{-2/11+\varepsilon}\,. \tag{2.43}$$

Since $qc_n < x^{1/11+1/12} < x^{2/11-\varepsilon}$ this leads to a contradiction.

We could get similar irrationality results if the functions $\sigma(n)$ or $\varphi(n)$ are replaced by $\sigma_k(n)(k \geqq 1)$ or products of powers of $\sigma_k(n)$ and $\varphi(n)$. In each case we would need the assumption that the a_n are monotonic, increasing faster than a certain fractional power of the numerators.

From Lemma 2.29 it is clear that there is a set of power $2^{\aleph_0}$ of series (2.25) which are rational even if we restrict the integers c_n to the values 1 or 2 since for $c_n = 1$ we can choose $a_n = \sigma(n) - 1$ or $\sigma(n) - 2$ to get $c_{n+1} = 1$ or 2 respectively and for $c_n = 2$ we choose $a_n = [(\sigma(n)\text{-}1)/2]$ to get $c_{n+1} = 1$ if $\sigma(n)$ is odd and $c_{n+1} = 2$ if $\sigma(n)$ is even. For the series with numerators $\varphi(n)$ we would have to use $c_n = 1, 2$ or 3 since all $\varphi(n)$ are even for $n > 2$.

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